Phys 325 Discussion 4[1] – The Space Shuttle

Last week, we began our study of multi-particle mechanics with a two-particle system: a rocket and its exhaust. We used the <u>general</u> form of Newton's second law, $\vec{F} = d\vec{p}\,/\,dt$ (as opposed to the special form, $\vec{F} = m\vec{a}$, that only applies when mass is constant), to derive the master differential equation for rocket motion:

$$\boxed{m\dot{v} = -\dot{m}v^{\text{ex}} + F^{\text{EXT}}}$$

- m and v are the mass and speed of the rocket; m includes any fuel the rocket has on board.
- v^{ex} is the speed of the exhaust = the fuel ejected out the back of the rocket. It is a design parameter of the rocket so it is always measured relative to the rocket's frame of reference.
- The term $-\dot{m}v^{\text{ex}}$ is called **thrust** and has units of force.
- F^{EXT} is the sum of forces of external origin (e.g. gravity, air resistance) that do not arise from the rocket's primary means of locomotion (recoil against the expelled fuel).

One final but very important note: The above equation describes 1-dimensional motion only – along the rocket's direction of motion – so that "F^{EXT}" is a bit disturbing: written like that, it looks like the *magnitude* of an external force ... but it is not! It is a force **component**. The $\dot{v}$ is similarly disturbing: magnitude or component? The distinction is important because of **signs**:

Vector <u>magnitudes</u> are always positive, but vector <u>components</u> can be positive or negative.

When you're working in 1 dimension, it is annoying to carry component-subscripts around all the time, but honestly we really should or we could easily make a sign error. Let's rewrite our rocket equation in this fashion. First, define $+x$ to be the direction in which the rocket is pointing. The exhaust will always emerge in the $-x$ direction relative to the rocket, and the thrust it creates will always push the rocket forward = toward $+x$. The rocket equation in careful, component form is then

$$m\dot{v}_x = -\dot{m}v_{\text{ex}} + F_x^{\text{EXT}}$$

Since v_x can be negative, we can now accommodate without confusion a situation where, e.g., the external force overwhelms the thrust and causes the rocket to move backwards.

Q1. Shuttle Launch

≈ Taylor 3.7 & 3.9; Checkpoints [2]

The first couple of minutes of the launch of a space shuttle can be described very roughly as follows: The initial mass including fuel is 2×10^6 kg, the final mass (after 2 minutes, when the fuel is gone) is about 1×10^6 kg, the average exhaust speed relative to the rocket is $v^{\text{ex}} \approx 3{,}000$ m/s, and the initial velocity is zero.

(a) If all this were taking place in outer space with no gravity, what would be the shuttle's final speed?

(b) Assuming that the rate $-\dot{m}$ at which the fuel is ejected is a constant, what is the thrust during these two minutes and how does it compare with the shuttle's initial total weight (on Earth)?

(c) Now add in the gravitational field g at the surface of the Earth. What is the minimum exhaust speed, $v_{\text{ex}}^{\text{min}}$, for which the shuttle would just begin to lift as soon as fuel burn is underway? (Keep all other design parameters unchanged.) Hint: The thrust must at least balance the shuttle's weight.

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[1] The assignments are numbered by **week** — there was no discussion #3 because of the holiday last week.

[2] Q1 (a) 2,100 m/s (b) $F_{\text{thrust}} = 2.5 \times 10^7$ N, $F_{\text{grav}} = 2.0 \times 10^7$ N (c) 2,400 m/s

Q2. Shuttle vs Gravity *inspired by Taylor 3.11; Checkpoints* [3]

The space shuttle takes off vertically from rest in the Earth's gravitational field g.

(a) Write down the master differential equation for the shuttle's motion in gravity by including the appropriate quantity for F_{EXT}. Remember: both the magnitude and *sign* of this term are important.

(b) Assuming that the rocket ejects mass at a constant rate $-\dot{m} = k$ (where k is a positive constant), solve your differential equation for v as a function of t. Self-check: $v(t) = v^{\text{ex}} \ln\left(\dfrac{1}{1 - kt/m_0}\right) - gt$

FYI: Note how this expression is arranged in the form recommended in Phys 225: quantities are grouped into **dimensionless terms** as much as possible (i.e. in the argument of that logarithm – the only place we can do it). You will see the advantages of this form in part (d)!

(c) Using the shuttle data from the Q1, find its speed two minutes into flight, assuming that it travels vertically upward during this period and that g doesn't change appreciably. (Both of these assumptions are excellent approximations.) Compare with the corresponding result when no gravitational force is present.

(d) The right-hand side of your master differential equation from (a) has two terms: the thrust term and the gravitational term. Describe what would happen to a rocket that had an initial velocity v_0 (instead of zero) but was designed so that the thrust term was smaller than the initial value of the gravitational term. Hint: To figure it out, use 1$^{\text{st}}$-order Taylor approximations (see footnote[4]) to determine the approximate behavior of $v(t)$ from part (b) for small times t, right after the engines kick in. "But what is small", you ask? Perfect question! The answer is supplied by the elegant dimensionless-term form we used in part (b): we see clearly that the logarithm's behaviour is controlled by the comparison between t and the **timescale** m_0/k. The small-time approximations you want are thus for times when $kt/m_0 \ll 1$.

(e) After that nice exercise in approximation, return to the normal situation when the shuttle starts from rest and the thrust force exceeds the gravitational force. Using your approximation skills and the part (b) formula, make a plot of the shuttle's speed vs time. Be sure to indicate the values (in symbols not numbers) of any important points on your graph, e.g. any starting or ending values, whether they're reached actually or asymptotically.

Q3. The Shuttle Climbs *inspired by Taylor 3.13*

(a) An engineer designs a new rocketship with parameters v_{ex}, m_0, and k. The engineer's intern uses those parameters to calculate $y(t)$ = the rocket's height as a function of time, then sends the result to you. Your job is to ensure the rocketship can make it from earth to the International Space Station in a certain amount of time (e.g. before the cockpit runs out of air — that's why you need to know $y(t)$!) The intern sends you this:

$$y(t) = v^{\text{ex}} t - \frac{1}{2} g t^2 - \frac{m v^{\text{ex}}}{k} \ln\left(\frac{m_0}{m}\right) \qquad \text{with} \qquad v^{\text{ex}} = 3\times10^3 \text{ m/s}, \quad m_0 = 2\times10^6 \text{ kg}, \quad k = 8.3\times10^3 \text{ kg/s}$$

Are you satisfied with this email? What email might you send *back* to the intern? ☺

(b) Using the $v(t)$ formula from Q2(b), calculate $y(t)$ yourself and confirm that the intern at least didn't make any errors in calculation. You'll need this integral: $\int \ln(x)\,dx = x\ln(x) - x$

(c) Using the data from the intern (same as from Q1), show that the ship's height after 2 minutes is ≈ 40 km.

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[3] Q2 (c) 900 m/s with gravity, 2,100 m/s without (d) $v(t) - v_0 \approx (v^{\text{ex}} k / m_0)t - gt < 0 \rightarrow v$ diminishes (e) check plot with your TA

[4] Here is the **Taylor Collection** from the Phys 225 1Dmath formula sheet: the five most useful leading-order Taylor approximations for small $\varepsilon \ll 1$: • $(1+\varepsilon)^n \approx 1+n\varepsilon$ • $\sin(\varepsilon) \approx \varepsilon$ • $\cos(\varepsilon) \approx 1-\varepsilon^2/2$ • $\ln(1+\varepsilon) \approx \varepsilon$ • $e^{\varepsilon} \approx 1+\varepsilon$ They'll save your life someday. ☺