Phys 325 Discussion 5 – Orbit and Spin

Here are the key formulae we proved for particle collections (also provided at the beginning of Homework 4):

- P - total momentum $\quad \vec{P} = M\dot{\vec{R}} \quad \vec{F}^{\text{EXT}} = \dot{\vec{P}} = M\ddot{\vec{R}} \quad$ with CM definition $M\vec{R} \equiv \sum_i m_i \vec{r}_i$
- L - total angular mom. $\quad L_\omega = I_\omega \omega \quad \vec{\tau}^{\text{EXT,(A)}} = \dot{\vec{L}}^{(\text{A})} \quad$ with definition $I_\omega \equiv \sum_i m_i r_{\perp i}^2$

<u>Notation</u>: **capital letters** denote **collective properties**, and sub/super-scripts have specific meaning:

$\vec{P}, \vec{L}, \vec{F}, \vec{\tau}, M$	**TOTAL** momentum, angular momentum, force, etc of/on the collection
$\vec{R}, X, \vec{V}, \vec{A}$	position, velocity, or acceleration of the collection's **CENTER OF MASS**
subscript	means "**OF**" $\rightarrow$ $\vec{r}_A$ = position OF point A … $\vec{L}_{\text{CM}}$ = angular mom. OF the CM
superscript ()	means "**RELATIVE TO**" $\rightarrow$ $\vec{\tau}^{(\text{A})}$ = torque RELATIVE TO POINT A $= \vec{r}_A \times \vec{F}$
superscript $'$	means "**RELATIVE TO the CM**" $\rightarrow$ $\vec{L}' \equiv \vec{L}^{(\text{CM})}$

Today's job is to add an extremely useful result: L (total) = L_{CM} (of the CM) + L' (around the CM). In symbols:

$$\vec{L} = \vec{L}_{\text{CM}} + \vec{L}' = \vec{R} \times \vec{P} + \sum_i \vec{r}_i{}' \times \vec{p}_i{}'$$

Note the use of **primes** as a convenient shorthand for "**relative to the CM**". The most evocative example of this decomposition is the motion of the earth around the sun. The earth has two types of angular momentum: it <u>orbits</u> around the sun and it <u>spins</u> around its own axis. Those *are* the two terms in our decomposition:

- Suppose our particle collection is the Earth and the origin of our coordinate system is the Sun.
- $\vec{L}_{\text{CM}} = \vec{R} \times \vec{P}$ describes the motion of the CM around the origin ➔ center-of-Earth around Sun = **Orbit**
- $\vec{L}' = \sum_i \vec{r}_i{}' \times \vec{p}_i{}'$ describes the motion of the particles around the CM ➔ Earth around its center = **Spin**

Q1. Prove *Hints & Checkpoints* [1]

<u>Prove</u> that the total $\vec{L} = \sum_i \vec{r}_i \times \vec{p}_i$ of a system around the origin can be written $\vec{L} = \vec{R} \times \vec{P} + \sum_i \vec{r}_i{}' \times \vec{p}_i{}'$.

(a) *Draw! A! Sketch!* Draw an origin, a random collection of particles, a center-of-mass position, and the CM vector $\vec{R}$. Then pick one of your masses, call it m_i, and draw the vectors $\vec{r}_i$ and $\vec{r}_i{}' \equiv \vec{r}_i - \vec{R}$ that point to it.

(b) This relation is key to our proof: $\sum_i m_i \vec{r}_i{}' = 0$. Either prove it mathematically, or write a sentence that convinces you of its obviousness. Your sentence might go something like this: "The sum on the left indicates the _____ relative to _____ , which is clearly ____"). Staring at the sketch will help. That always helps. ☺

(c) Go go go! PROVE that $\vec{L} = \vec{R} \times \vec{P} + \sum_i \vec{r}_i{}' \times \vec{p}_i{}'$.

(d) Finally, we need <u>torque laws</u> that tell us how <u>each term</u> of our decomposition changes with time.
For L (**OF** the CM), show that $\dot{\vec{L}}_{\text{CM}} = \vec{R} \times \vec{F}^{\text{EXT}}$ = total torque on the CM, just as you would expect.

For L (**AROUND** the CM), we already have an equation for $\dot{\vec{L}}'$ … look up top, and write down what it is.

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[1] (b) "… indicates the *position of the CM* relative to *the CM*, which is clearly *zero*"

(c) Hint: decompose the $\vec{r}_i$'s in $\vec{L} = \sum \vec{r}_i \times \vec{p}_i$ to get $\vec{L} = \sum (\vec{R} + \vec{r}_i{}') \times m_i (\dot{\vec{R}} + \dot{\vec{r}}_i{}')$ … (d) $\dot{\vec{L}}' \equiv \dot{\vec{L}}^{(\text{CM})} = \vec{\tau}^{\text{EXT,(CM)}}$

Q2. Play in a Sandbox *Checkpoints* [2]

To intuitively understand what we've proved, we need a **sandbox**: a familiar system that we can play with that incorporates both an "orbital" part and a "spin" part. Here is a good 2D sandbox for experimenting: two masses moving in the xy plane with the y axis pointing up and uniform gravity g pointing down.

Let's use two equal point-masses m=1. The left one is placed at $(x, y) = (-b, h)$ and dropped from rest; it hits the ground (y=0) at time T. The right-hand mass is placed at $(x, y) = (+b, 0)$ and thrown vertically upward; its initial velocity is chosen so that at time T when the left mass hits the ground, the right mass reaches the height h.

(a) As our given parameters, let's use b, h, and T — an unusual choice, but it makes the upcoming expressions more elegant, trust me. Before you do anything else, calculate v_0 (initial velocity of the right mass) and g (gravitational acceleration) in terms of h and T and put them in boxes → we'll need those relations.

(b) We know the solution to this problem! That's why it's a good sandbox. ☺ Write down the trajectory $x_L(t), y_L(t)$ of the left mass and $x_R(t), y_R(t)$ of the right mass, then determine their velocities $\vec{v}_L(t)$ and $\vec{v}_R(t)$.

(c) Now the **Center of Mass**! Where is it? Write down its trajectory $X(t), Y(t)$ and its velocity $\vec{V}(t)$.

(d) Make a <u>plot</u> of the CM's height vs time, $Y(t)$, and its velocity $V(t)$. Take as much time with these plots as you like to convince yourself that you understand the motion of the CM.

(e) The CM is supposed to behave as if the total mass M of the system is located at the CM position and is subject to the total external force on the system: $\dot{\vec{P}} = \vec{F}^{\text{EXT}} = M\ddot{\vec{R}}$. Evaluate $\vec{F}^{\text{EXT}}$ & $M\ddot{\vec{R}}$ and see if they match.

(f) On to **Angular Momentum around the Origin**: calculate the total $\vec{L}$ of our 2-particle collection.
For your convenience, when $\vec{r}$ is in the xy-plane and $\vec{v}$ is in the y direction only: $\vec{r} \times \vec{v} = \hat{z}\, x v_y = \hat{z}\, x\dot{y}$.

(g) The total angular momentum is supposed to obey the elegant collective relation $\dot{\vec{L}} = \vec{\tau}^{\text{EXT}}$ that reflects the cancellation of internal forces (just like $\dot{\vec{P}} = \vec{F}^{\text{EXT}}$). What is the total $\vec{\tau}^{\text{EXT}}$? Convenience: $\vec{r} \times F_y \hat{y} = \hat{z}\, x F_y$

(h) That was the left-hand side of $\vec{L} = \vec{R} \times \vec{P} + \sum_i \vec{r}_i{}' \times \vec{p}_i{}' = \vec{L}_{\text{CM}}(\text{orbit}) + \vec{L}'(\text{spin})$. Now the right-hand side. Is there any "**Orbital**" part here? → What is the angular momentum $\vec{R} \times \vec{P}$ of the CM around around the origin?

(i) Now the "**Spin**" part: calculate the total $\vec{L}'$ around the CM and verify that the relation holds!

Q3. Shift *Checkpoints* [3]

One of the most confusing aspects of angular momentum is how it changes depending on where we place the origin. It is very important to understand this! ➔ Let's **shift our origin to the left** by a distance b. The masses will now be moving along the lines $x_L = 0$ and $x_R = 2b$, and the CM will move along $X = b$. The $y(t)$ vertical trajectories of the masses and the CM will be unchanged ... but the angular momenta *will* change. Let's see!

(a) Calculate the total $\vec{L}$ and $\vec{\tau}^{\text{EXT}}$ around our new origin and verify that $\dot{\vec{L}} = \vec{\tau}^{\text{EXT}}$.

(b) Calculate the orbital motion of the CM, $\vec{L}_{\text{CM}} = \vec{R} \times \vec{P}$, around our new origin. Has spin=$\vec{L}'$ changed? (No.)

(c) Finally calculate the torques $\vec{\tau}_{\text{CM}}^{\text{EXT}}$ and $\vec{\tau}'^{\text{EXT}}$, and check their relations to $\vec{L}_{\text{CM}}$ and $\vec{L}'$.

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[2] (a) $g = 2h/T^2$, $v_0 = 2h/T$ (b) $x_R(t) = b$, $y_R(t) = h\,(2t/T - t^2/T^2)$, $x_L(t) = -b$, $y_L(t) = h\,(1 - t^2/T^2)$ (c) $X(t) = 0$, $Y(t) = h\,(\tfrac{1}{2} + t/T - t^2/T^2)$ (e) both = $-2mg\,\hat{y}$ (f) $\vec{L} = \hat{z}\; 2bh/T = \text{constant}$ (g) $\vec{\tau}^{\text{EXT}} = 0$ (h) $\vec{L}_{\text{CM}} = 0$ (i) $\vec{L}' = \hat{z}\; 2bh/T$

[3] (a) $\vec{L} = \hat{z}(4bh/T)(1 - t/T)$, $\vec{\tau} = -\hat{z}\,4bh/T^2$ (b) $\vec{L}_{\text{CM}} = \hat{z}(2bh)(1/T - 2t/T^2)$, $\vec{L}' = \hat{z}\; 2bh/T$ (c) $\vec{\tau}_{\text{CM}} = -\hat{z}\,4bh/T^2$, $\vec{\tau}' = 0$