Phys 325 Discussion 6 – Rollers, Strings, and Pulleys

Many rigid body problems include **rolling objects**, **massless connectors**, and/or **strings & pulleys**. Working with these objects requires some thought. Specifically, each of these three object classes brings with it a condition that you must usually apply to solve the problem:

(1) Rolling objects: apply **no-slip condition** (usually $\omega = V/r$) to relate rotation speed to CM speed.

(2) Massless connectors: apply **zero net force** condition on any massless object.

(3) Strings and pulleys: apply **conservation of string length** condition.

The first condition is covered in lecture. The other two need some comment and some examples.

Problem 1: Explanation of Techniques (2) and (3) *Checkpoints* [1]

Many mechanics problems feature massive objects that are connected by "massless" chains, support beams, strings, etc. Of course "massless" means "of negligible mass compared to everything else", but the point is that you can treat such objects as having zero mass. Here's how:

The total external force on any massless object must be zero.

Why? Because $\vec{F}^{\text{EXT}} = M\ddot{\vec{R}}$, so if an object has zero total mass and *non*-zero total external force, it will have infinite acceleration! That is a non-physical situation, so we must ensure that $F^{\text{EXT}} = 0$ on any massless object.

In the figures below, a dog and a cat are each trying hard to move in the direction they're facing. On the left, they're connected by a massless string and putting it under tension (= "stretching" force); on the right, they're connected by a massless bar and putting it under compression (= "squeezing" force). The arrows turn the figures into free-body diagrams (**FBD**s) for the connectors, i.e. diagrams that show all the forces acting on the string and the bar. With our arrows and our labeling, we ensure that the forces on either end are equal and opposite, and this imposes the **zero net force** condition on our massless objects. Here's the result:

Free-Body Diagrams for string and bar

(a) It is likely that we are interested in the motion of the *massive* objects – the dog and the cat – rather than the connectors. To determine the animals' motion, we need *different* FBDs: ones that show the external forces acting *on each animal*. First, the cat and the dog are trying hard to move forward; they push against the floor, and the responding force of friction pushes them in the direction they want to move. The dog, being stronger, gets a force of $F_d = 50$ N from the floor, pointing to the right. The smaller cat digs its claws into the ground but gets only $F_c = 20$ N from the floor, pointing to the left. The masses of the dog and cat are $m_d = 10$ kg and $m_c = 2$ kg respectively. Use arrows & labels to turn the pictures below into FBDs for the dog and cat.

Free-Body Diagrams for cat and dog

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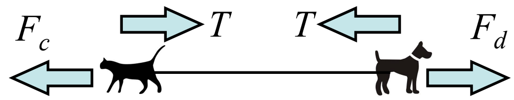
[1] (a) Each animal should have two force arrows attached to it: (1) the friction force = F_c to the left for the cat and F_d to the right for the dog, and (2) the force from the attached connector = T pointing *inward* — i.e. toward the center of the string — for the left-hand figure and N pointing *outward* for the right-hand figure. Flip the page for the string diagram. (b) cat: $T - F_c = \ddot{x}_c$, dog: $F_d - T = \ddot{x}_d$ (c) $\ddot{x}_d = \ddot{x}_c$ (d) $T = \dfrac{F_d m_c + F_c m_d}{m_c + m_d} = 25$ N (e) $\ddot{x}_{\text{string}} = \ddot{x}_d = 2.5$ m/s^2 (f) FBD has only 2 forces: F_c and F_d; $\ddot{X} = \dfrac{F_d - F_c}{m_d + m_c} = 2.5$ m/s^2

(b) Let's now focus on the string case. Define the $+x$ direction to point to the right, then write down the equations of motion you need to determine the animals' accelerations $\ddot{x}_d$ and $\ddot{x}_c$.

FBDs for
cat & dog



(c) We have too many unknowns to solve for $\ddot{x}_d$ and $\ddot{x}_c$ yet → we need another relation. It's time to introduce the other technique on today's menu: **conservation of string length**. In almost all mechanics problems with strings or chains, they are assumed to behave like

> **Ideal Strings** ≡ objects that are infinitely flexible (offer no resistance to being bent around corners or crumpled into piles) but whose length never changes. No matter how much tension they are under, they never stretch. For a bit more info beyond what we need today, see footnote[2].

The length property is what we need today: for strings under tension, their total length is fixed, and that's a condition we *must* apply to solve string problems. The string between the cat and the dog is certainly under tension; use the fact that its length is fixed to write down an *absolutely obvious* but absolutely essential relation between $\ddot{x}_d$ and $\ddot{x}_c$.

(d) *Now* you can solve the problem! But before you do, *make a guess*: given the values $F_d = 50$ N, $F_c = 20$ N, $m_d = 10$ kg, and $m_c = 2$ kg, what do you think the string tension T is? ... 70 N? ... 30 N? something else? ... Once you've made your guess, use your two equations of motion and the string-length condition to calculate the string tension. Surprised? ☺ If so, see footnote[3].

(e) What is the acceleration of the string? If you are surprised at the answer, see footnote[4].

(f) So far, we've drawn FBDs for three different "free bodies": the string, the dog, and the cat. You can also draw FBDs for **subsystems** = combinations of connected objects. This technique can be very helpful to isolate particular forces or motions! The most common example is to analyze the **entire system** and treat the [cat + string + dog] system as your free body. What are the external forces acting on [cat + string + dog]? Draw and label them on an FBD and write down the corresponding EOM for the entire system.
Your EOM should give you the acceleration $\ddot{X}$ of the center of mass, and it should make sense.
Reminder: FBDs never show forces that are *internal* to your chosen free-body, only *external* ones. The reason for that is the wonderful cancellation of internal forces that led to our elegant collective-EOM, $\vec{F}^{\mathbf{EXT}} = \dot{\vec{P}}$.

One final comment about **massless connectors**: If you apply the zero net force condition to a small segment of a massless string, you immediately see that the tension at either end has to be the same … and since you can do that for *every* tiny segment along the string, you immediately get this often-quoted result:

Tension has the same magnitude everywhere along a massless string.

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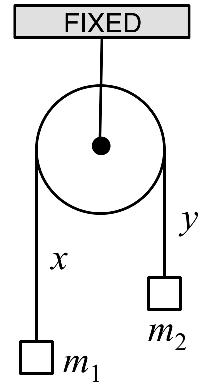
[2] FYI: An ideal string doesn't have to be massless. If it has mass, an excellent way to model it is as a chain of tiny beads connected by massless ideal strings. That's beyond today's material, but is a very useful thing to know.

[3] Are you surprised that the string tension is not 70 N? Well so am I! ☺ I honestly find string tension to be the most *utterly non-intuitive* concept in mechanics. Hopefully your physical intuition is better, but please note how complex this thing is: it is *not* just the result of the external forces on the two ends → it also depends on the final acceleration of the objects, and therefore on the masses involved. The best way to gain intuition is to play with your tension formula from part (d) and the upcoming acceleration formula in part (f) to see what they does in various limiting cases, e.g. when $m_d >> m_c$ or when $F_d = F_c$.

[4] Are you surprised that the string's acceleration is not zero? We carefully set things up so that the *net force on the string is zero* … yet clearly the dog, cat, and string are all accelerating at the same non-zero rate. Reason: since $F = ma$, an object with no mass doesn't *need* any force to accelerate!

Problem 2 : Atwood's Machine *Checkpoints* [5]



An Atwood machine is a simple device that was originally devised to measure g. It consists of two masses suspended at the ends of a string, with the string running over a massless, frictionless pulley whose axle is fixed. There is only one relevant spatial dimension in this problem: vertical position below the pulley's axle.
The figure labels this position x for mass m_1 and y for mass m_2. These positions are not independent, however, because of **conservation of string length**.

(a) Calculate the acceleration $\ddot{x}$ of the mass m_1 using:

- two free-body diagrams (and their associated $\vec{F}^{\text{EXT}} = d\vec{P}/dt$ equations of motion, of course → that is the *point* of drawing FBDs!), and
- the conservation-of-string-length condition.

About the string length condition:

- If you're wondering how you can apply this condition when you don't *know* the length of the string, the answer is *you don't need to know it*. Just give it a label, like l. All you need to know is that l is constant.
- String length always leads to a **<u>relation between different accelerations</u>** <u>and/or velocities</u> in the system. In the dog/cat problem, string length gave you $\ddot{x}_c = \ddot{x}_d$ → the fixed length of the string forced any change in the cat's position to be exactly the same as for the dog. If you had needed a velocity relation instead (less common, but sometimes you do) it would have been $\dot{x}_c = \dot{x}_d$. Here, string length will relate $\ddot{x}$ to $\ddot{y}$.

This is a warm-up problem. If you're not done in 5 minutes, read the hints in the checkpoint.

(b) When you applied conservation of string length, did you write $l = x + y$? If so, you ignored the part of the string that runs over the top of the pulley. Oh no! Must you go back and change your calculation?
Answer: NO. You did exactly the right thing … but is it clear WHY?

(c) Draw a FBD for the massless pulley and use it to obtain a relation between the string tension T and the normal force F_N exerted by the fixed ceiling. Hint 1: Segments of a massless string are also massless.
Hint 2: A massless pulley has zero mass … and therefore zero moment of inertia; given that $\tau = I\dot{\omega}$, where $\dot{\omega}$ is angular acceleration, what does that tell you about the **net torque on a massless pulley**?

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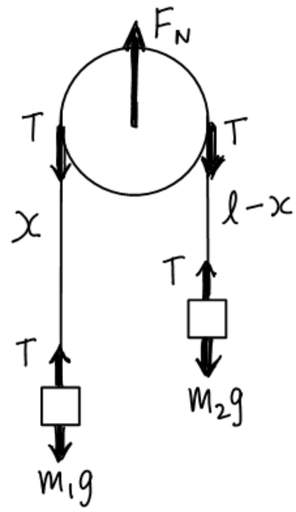
[5] (a) <u>Hint #1</u>: The only strategic issue here is which FBDs to draw. You can analyze the forces on any of the following "free bodies": mass 1, mass 2, the pulley, the string, the entire system, or any connected subset of objects. Here's the best tactic: try to **avoid normal or tension forces** that you don't know and don't care about. In this problem you can't avoid the string tension, since the string is connected to the mass whose acceleration you want … but you *can* avoid the normal force between the fixed platform and the pulley. The best free bodies to choose are m_1 and m_2.
<u>Hint #2</u>: When drawing the FBDs for m_1 and m_2, remember that the tension forces at the ends of the string are equal in magnitude & opposite in direction … but hold on, this string *bends* … what does "opposite in direction" mean here? → Think of it this way: objects under tension **pull on both ends**, while objects under compression **push on both ends**.
<u>Hint #3</u>: If you are still unsure about how to draw those FBDs, flip the page.
<u>Answer</u>: $\ddot{x} = g(m_1 - m_2)/(m_1 + m_2)$

(b) The string segment that runs over the pulley is of **constant length** (π × pulley-radius) so it doesn't affect the fact that $x + y =$ *constant*, which is all that matters: your *goal* is to get a relation between $\ddot{x}$ and $\ddot{y}$, not between x and y. You can ignore *any* segment of constant length: just define your "l" to be the total length of the string *minus* any constant-length pieces.
(c) $F_N = 2T$. Explanation: see the top part of the 3-in-1 free-body diagram on the next page. It shows that the net force on the pulley is $F_N - 2T$, and since the pulley is **masssless**, this net force must be zero. About Hint 1: the vertical string segments have equal-and-opposite tensions T at their top and bottom ends because they are massless. About Hint 2: the torque-inducing tensions T on either side of the pulley have to be the same because the **total external torque on an object with zero moment of inertia must be zero** (otherwise you would have infinite angular acceleration). This is the angular analogue of "zero net force on a massless object".

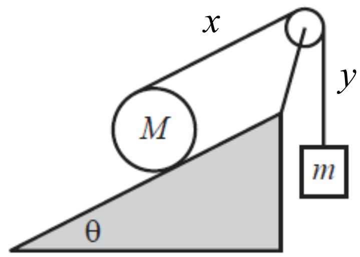
Problem 3 : A Pulley and a Ramp *Checkpoints* [6]

Now let's combine our techniques for rolling objects, massless connectors, and strings in one problem. Below you see a solid cylinder of mass M and radius r rolling on a ramp of slope θ. A massless string is wound around the cylinder, then runs up and over a pulley, and is finally connected to a suspended cube of mass m. Uniform gravity g points down. Reminder: the moment of inertia of a solid cylinder for rotation around its axis is $I = Mr^2/2$.

3-in-1 FBD for Atwood machine



You have one task: calculate the acceleration $\ddot{y}$ of the cube.



You can try to solve this without any assistance, but it's a complex problem that requires 5 different equations. If you would like some guidance, follow the steps below.

(a) First, we need FBDs for both the cylinder and the cube, so draw and label all the forces acting on them. You must introduce labels for two unknown forces: use T for the string tension and f for the frictional force between the cylinder and the ramp. (There is also a normal force between the cylinder and the ramp, but we will be able to avoid it.) Also add the label ω to your diagram to denote the rotational speed of the cylinder, and be careful to pick a *direction* for it → let's choose counter-clockwise (so that positive ω means x is increasing). Show this choice with an arrow.

(b) You have five unknowns: x, y, ω, T, f , all of which may be time-dependent. You thus need five differential equations to solve the problem.[7] The first three are familiar force / torque laws, $\vec{F}^{\text{EXT}} = \dot{\vec{P}}$ or $\vec{\tau}^{(A)} = \dot{\vec{L}}^{(A)}$. Different choices are possible but I suggest these three:

1. force on cube 2. force on cylinder: component parallel to ramp 3. torque on cylinder around its CM

Set up these equations in terms of the givens m, M, r, θ and the unknowns x, y, ω, T, f (or their t-derivatives).

(c) The 4th equation is the **no-slip condition** on the rolling cylinder.

(d) Now do some algebra: combine equations 1,2,3,4 to get rid of all unknowns except x and y. (It's just algebra, you can just grab the answer from the checkpoint if you are confident of your algebra skills.)

(e) The last equation is **conservation of string length** … and it is a *tough* one because string is "removed" from the system as it winds around the rolling cylinder. (That's assuming ω is positive; if ω is negative, string is "added" to the system, but no worries: that sign will take care of itself. Always treat variables as positive when deciding on the signs to put in your equations). Two strategies are given in the checkpoint.

(f) Get the final answer: what is the acceleration of the cube?

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[6] (b) Eq#1: $m\ddot{y} = mg - T$ Eq#2: $M\ddot{x} = Mg\sin\theta + f - T$ Eq#3: $T + f = -Mr\dot{\omega}/2$ (c) Eq#4: $r\omega = \dot{x}$ or $r\dot{\omega} = \ddot{x}$

(d) $4m\ddot{y} - 3M\ddot{x} = 2g(2m - M\sin\theta)$ (e) Strategy 1: Let $l = x+y$ be the amount unrolled string. Then dl/dt is the rate at which string is unrolled from the cylinder, and that's proportional to ω and r … think carefully, and you'll see that dl/dt is $-r\omega$. Strategy 2: Put an imaginary dot of paint *on the string* at the point it touches the cylinder. The speed v of that dot up the ramp is the same as the cube's downward speed $\dot{y}$ … and the dot's speed v is the same as that of the *top* of the cylinder relative to the ramp ... which is $-2r\omega$.

Answer: Eq#5 is $2\dot{x} = -\dot{y}$ or equivalently $2\ddot{x} = -\ddot{y}$ (f) **Final Answer**: $\ddot{y} = g\left(2m - M\sin\theta\right)/\left(2m + \frac{3}{4}M\right)$

[7] Do we really need five equations? Not necessarily → it *is* possible to be clever and *avoid* some of your unknowns. In fact we already did: we have a 6th unknown = the normal force from the ramp, but our procedure is going to avoid it. So we have already reduced the needed number of equations from 6 to 5. I can't find a way to get it below 5, but maybe you can!