Phys 325 Discussion 7 – Small Oscillations & Equilibrium

Here is a phrase that pops up all over the place: **Small Oscillations**. It is closely connected to the notions of **equilibrium** and stable-vs-unstable equilibrium. Let's investigate!

- If a system is **oscillating**, we mean it "moves back and forth", perpetually returning to the same position.
- If the system keeps returning to the same position, there must be a **restoring force** that pulls it toward some **equilibrium position** that it passes through repeatedly (or circles around, in 2D or 3D problems).
- Let's introduce some generic coordinate ***x*** to describe the state of our system, and let $\boldsymbol{x = 0}$ denote the system's equilibrium position. (This x could be a position or an angle, or even a set of coordinates if our system has more than one dimension; the point is that x denotes the state of the system.)
- The "**small**" part of "small oscillations" means we are restricting ourselves to motions where the coordinate x remains **close to the equilibrium position $\boldsymbol{x=0}$**. If x is always small, the restoring force $F_x(x)$ can be fruitfully approximated with a **Taylor series** around x=0:

$$F_x(x) \approx F_x\big|_{x=0} + x\frac{dF_x}{dx}\bigg|_{x=0} + \frac{x^2}{2!}\frac{d^2F_x}{dx^2}\bigg|_{x=0} + \text{negligible}$$

- The **restoring force** $\vec{F}$ keeps pulling the system toward the equilibrium position $x = 0$. (Otherwise the system wouldn't oscillate!) The force component F_x must therefore be an **odd function of *x*** :
  - when $x > 0$, F_x must be negative to pull the system back toward $x = 0$
  - when $x < 0$, F_x must be positive to pull the system back toward $x = 0$

  The constant and quadratic terms in the Taylor expansion must therefore be zero, and the coefficient of the linear term has to be negative. (Another reason the constant term has to be zero is because x=0 is an equilibrium point, so there is *zero force* there.) Result:

$$F_x(x) \approx -kx \qquad \text{where the constant } k \equiv -\frac{dF_x}{dx}\bigg|_{x=0} \text{ is positive}$$

- So : if we stick to small enough x, our restoring force must have an approximately **linear** dependence on x. A linear force with a negative constant? We've seen that before → that's Hooke's Law for a linear spring. If you inject $F_x(x) \approx -kx$ into $F_x = m\ddot{x}$, you get this super-familiar equation of motion:

$$\ddot{x} = -x\ (k/m).$$

  We know the general solution to that: $x(t) = A\cos(\omega t) + B\sin(\omega t)$ or $x(t) = C\cos(\omega t + \phi)$, with $\omega = \sqrt{k/m}$. This ω is called the **frequency of small oscillations** of the system.

☞ Many problems in physics and engineering end with the phrase "what is the frequency of small oscillations?" When you see "small oscillations", you must do this :

Approximate the restoring force for **small deviations of the system from equilibrium** using a **leading-order Taylor approximation**.

You will not be *told* to make this approximation, it is *implied* by the phrase "small oscillations". Let's tackle some questions of this type involving rotations. In all of them, uniform gravity is active and provides the restoring force.

Problem 1 : Small Oscillations for a Compound Pendulum *Checkpoints* [1]

A rod of length d and mass M has one of its ends attached to a fixed pivot that allows it to swing freely under the influence of gravity. The rod has a non-uniform mass distribution that results in a moment of inertia I for rotation around its end (along any axis perpendicular to the rod) and a center-of-mass position that is a distance R from the end that's attached to the pivot. The given quantities are thus d, M, I, R, and of course g.

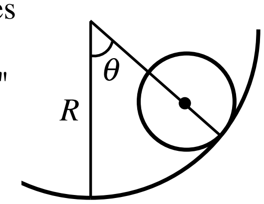
(a) What is the frequency of small oscillations of the pendulum?

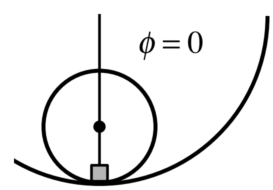
(b) You undoubtedly used the letter "**ω**" for the small-oscillation frequency. And now let's confront a **massive point of confusion** in rotational problems: *Is this ω the angular velocity of anything? i.e., Is it the time-derivative of any angle?* Answer: *no!* You can verify this yourself, but do ask your instructor if it's not 100% clear! This also carries a message: beware of the formula "$L=I\omega$" in small-oscillation problems … best advice is never to write it like that, but always as $L = I\dot{\theta}$ or $L = I\dot{\phi}$. Same for torque: write $\tau = I\ddot{\theta}$ not $\tau = I\dot{\omega}$ Then ω is reserved for the oscillation frequency. (If you prefer, you can use Ω for oscillation frequency.)

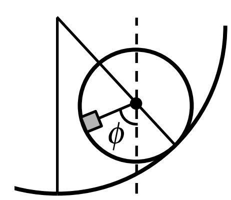
Problem 2 : Small Oscillations for a Ball in a Fixed Cylinder *Checkpoints* [2]

A small ball with radius r and uniform mass m rolls without slipping near the bottom of a fixed cylinder of radius R, where $R > r$. Our goal is to find the frequency of small oscillations of the small ball.

(a) The moments of inertia for a ball of radius b and uniform mass M for rotation around its center (= its CM) is $I' = 2Mb^2/5$. You may find it useful in this problem to also know the moment of inertia, $I^{(\text{edge})}$ for rotation of the ball around an edge-point, with the axis of rotation being tangential to the sphere. For practice, use the **parallel-axis theorem**, $I^{(B)} = I^{(B)}_{\text{CM}} + I'$ to find $I^{(\text{edge})}$ for the ball.







(b) The trickiest part of this problem is the **no-slip rolling condition**. This condition relates the rolling object's rotation angle ϕ to its CM position X. In this problem, the ball's CM position is most easily described by the angle θ shown in the top figure, so θ will be our "X" coordinate. The rolling condition must therefore relate the ball's rotation angle ϕ to its CM position θ … which means *two angles* are involved. Before you add ϕ to the sketch, you must think carefully about what **rotation angle** actually means. In words: mentally paint a marker somewhere on the ball's edge so you can tell its orientation; the ball's rotation angle ϕ is the angle the marker makes with the vertical axis, or any other fixed axis of our inertial frame. The lower two pictures at right illustrate this definition using the small grey square as the orientation marker. They show how the ball's orientation changes as it rolls without slipping. Your job is to combine these figures and come up with the no-slip rolling condition that relates ϕ to θ. (It will also involve r and R … hint: *arc length* ...)

(c) Now off you go: find the frequency of small oscillations of the ball! A hint[3] is available with detailed strategies if you are stuck.

(d) If you solved the problem without the hint, bravo! ☺ But the hint has some useful information : it describes multiple methods for solving this problem. For practice, please re-solve the problem using the other techniques.

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[1] Q1 (a) $\omega = \sqrt{MgR/I}$ (b) No, it's an angular **frequency**, $\omega = 2\pi f$, not an angular speed!

[2] Q2 (a) $I^{(\text{edge})} = 7Mb^2/5$ (b) $r\phi = (R-r)\theta$ (c,d) $\omega = \sqrt{5g/7(R-r)}$

[3] You can solve this in multiple ways: (1) with two equations : $F = MA$ tangential to the cylinder & $\tau' = I'\ddot{\phi}$ (2) with one torque equation around a well-chosen point that *zaps* the friction and normal forces (3) using conservation of energy, once you've convinced yourself that the normal and rolling-friction forces *do no work*; this method itself even has two versions if you consider that the ball's kinetic energy can be calculated in two different ways. For all methods, you also need $r\phi = (R-r)\theta$ from the previous hint.