Phys 325 Discussion 9 – Equilibrium & Energy

Equilibrium Analysis : Is U enough?

In class, we discussed two methods for finding a system's **equilibrium points** and determining their **stability**. For a system with one independent coordinate "q" the methods are:

- Method **A**: Analyze the first and second derivatives of the **potential energy** $U(q)$.
  - The equilibrium points $\bar{q}$ are the points where local extrema of U occur : $U'(\bar{q})=0$
  - The equilibrium is stable / unstable if the curvature of U is positive / negative : $U''(\bar{q})>0 \rightarrow$ stable
- Method **B**: Use the system's **EOM**(s) to find the equilibrium points, then **solve** the **Taylor-approximated** EOMs very near equilibrium in the style of a **small-oscillation analysis** to determine stability.
  - The equilibrium points $\bar{q}$ are the points where the EOM is solved by $q(t)$ = a constant $\equiv \bar{q}$.
  - The equilibrium is stable / unstable if the EOM's solution near $\bar{q}$ oscillates / runs away.

The U-based method A is *fast*, but it *doesn't always work*. In Method B you explicitly solve for the motion of the system near equilibrium, so that never fails. The conditions under which **Method A works** for energy-conserving problems with one independent coordinate q are derived in Homework 8. Here they are:

(1) Potential energy must have this form: $\boxed{U(q)}$ + optional irrelevant constant

(2) Kinetic energy must have this form: $\boxed{T(q,\dot{q})=\frac{1}{2}\mu(q)\,\dot{q}^2}$ + optional irrelevant constant.

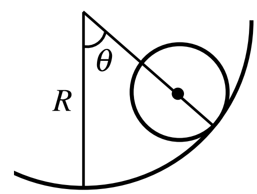
$\boldsymbol{\mu(q)}$ plays the role of an "**effective mass**" that may depend on the state of your system. A perfect example is the falling & sliding cube from Homework 7, whose kinetic energy was $T(\phi,\dot{\phi})=\frac{1}{2}\dot{\phi}^2\left(I'+\frac{1}{2}Mb^2\sin^2\phi\right)$. The term in parentheses is the cube's "effective mass" $\mu(\phi)$, which changes as the cube rotates.

(3) The effective mass $\boxed{\mu(q) \text{ must behave like a normal mass}}$ in these rather common-sense ways:

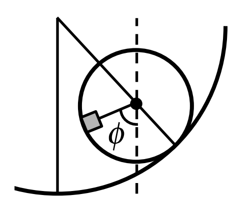
- $\mu(q)$ and its derivative $\mu'(q)$ must be finite (no singularities!) at all values of q.
- $\mu(q)$ must be positive at all equilibrium points $\bar{q}$.

If μ is just a constant, as in $T=\frac{1}{2}m\dot{x}^2$ or $T=\frac{1}{2}I\dot{\phi}^2$, these conditions are trivially satisfied.

Problem 1 : Ball in a Fixed Cylinder *Checkpoints* [1]



In Discussion 7 you analyzed the small oscillations of a ball of radius r that rolls without slipping near the bottom of a stationary cylinder of radius R. The figures define the two relevant angles in this problem, which are linked by the rolling condition $r\phi=(R-r)\theta$.



(a) In discussion 7, you used a combination of torques and forces to build the EOM. This time, **use energy**: the system conserves T+U, so use that to obtain the EOM. You should get the same differential equation as before: $\ddot{\theta}=-5g\sin\theta\,/\,7(R-r)$. See the footnote for a hint if you're having trouble calculating T.

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[1] (a) Hint: We want T relative to the origin, but the ball is not spinning around the origin, so we have to think about reference points. We have two reference-point formulae for T: One is $T=T^{(\text{stationary})}$ where (stationary) means any point that has *zero velocity* relative to the origin. The other is the usual CM-decomposition: $T=T_{\text{CM}}+T'$. Either one can be used here. Also, *whenever* you obtain an EOM from $\dot{T}+\dot{U}=0$, there will always be a velocity that is common to all terms and can be cancelled.

(b) $\Omega=\sqrt{5g\,/\,7(R-r)}$ is what you got before, and you should get it again. ☺ (c) $\bar{\theta}=0\ \rightarrow$ stable; $\bar{\theta}=\pi\ \rightarrow$ unstable … assuming we can reach θ=π at all, which would require something sticky like velcro to keep the ball in contact with the cylinder.

(b) Calculate the small-oscillation frequency Ω of the ball around $\theta = 0$.

(c) That was Method B. Now let's try Method A. Use only the potential energy U to find the system's equilibrium points $\bar{\theta}$ and to determine if each is stable or unstable.

(d) On Homework 8, you will derive a third result for Method A: near points $\bar{q}$ of stable equilibrium, the small-oscillation frequency is given by the extremely useful formula $\Omega = \sqrt{U''(\bar{q})/\mu(\bar{q})}$. (Around *unstable* points, there is no oscillation frequency since the system *doesn't oscillate* there, it runs away. ☺) Compare your expression for kinetic energy from part (a) to the boxed formula on the previous page to determine the effective mass $\mu(\theta)$, then use the formula to calculate Ω around this system's one point of stable equilibrium. It should match what you got in part (b). Method A is indeed quick and powerful!

Problem 2 : Bead on a Wire *Hints & Checkpoints* [2]

A bead of mass m is threaded onto a wire that lies in the xy-plane and has this shape: The bead moves without friction along the wire. Gravity is not present; instead, a mysterious force field exerts a force $\vec{F}(x) = -kx\hat{x}$ on the bead, where k is a positive constant. Does this system have any equilibrium points?

$$y = 3a\left(\frac{x}{a} - 1\right)^{1/3}$$

(a) Let's try our fabulous Method A. Using x as your coordinate, calculate the potential energy $U(x)$ of the bead. (You can ignore the normal force exerted by the wire because it is a *no-work* force: it never does any work, so it doesn't contribute to $U(x)$. Instead, its effect is entirely taken into account by the constraint equation above that relates y to x. The fact that such no-work constraint forces make no appearance at all in U is one of the many advantages of using an energy-based analysis over a force-based one. ☺) Once you have $U(x)$, figure out where the equilibrium point(s) $\bar{x}$ are, and whether their equilibrium is stable or unstable.

(b) Now let's try the ever-reliable Method B. First we need the bead's equation of motion. Energy is conserved for this frictionless system, so we can obtain our EOM using $T+U$ conservation. Calculate T in terms of x, $\dot{x}$, and the given constants k, m, and a.

(c) Now build your equation of motion, $\dot{T} + \dot{U} = 0$. The EOM will be fairly ugly, but not too bad.

(d) Now plug in the equilibrium solution $x(t) = \bar{x} =$ constant ... what equilibrium points $\bar{x}$ solve the EOM?

(e) Oh no! Method A has clearly failed! It completely missed one of the equilibrium points. First <u>sketch</u> the situation to satisfy yourself that Method B is the correct one, i.e. that both equilibrium points it found make sense. Then, figure out which condition on page 1 was violated to produce the failure of Method A.

(f) There is a surprisingly easy fix to this situation: *use y as your coordinate instead of x*. Rewrite your energies in the form $U(y)$ and $T(y,\dot{y})$, then verify that the troublesome condition that caused problems before is now perfectly fine. (Knowing which condition caused problems is exactly what allows you to figure out the fix!) Finally, apply Method A to your new $U(y)$ to find the two equilibrium points and determine their stability.

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[2] (a) $U(x) = \dfrac{kx^2}{2} \to$ 1 stable equilib point $\bar{x} = 0$ (b) $T = \dfrac{m\dot{x}^2}{2}\left[1 + \left(\dfrac{x}{a} - 1\right)^{-4/3}\right]$ (c) $\left(\ddot{x} + \dfrac{kx}{m}\right)\left(\dfrac{x}{a} - 1\right)^{7/3} + \ddot{x}\left(\dfrac{x}{a} - 1\right) = \dfrac{2}{3a}\dot{x}^2$

(d) $\bar{x} = 0, a$ (e) condition (3) bullet #1 (f) $U = \dfrac{ka^2}{2}\left[1 + \left(\dfrac{y}{3a}\right)^3\right]^2$, $T = \dfrac{m\dot{y}^2}{2}\left[1 + \left(\dfrac{y}{3a}\right)^4\right] \to$ $\bar{y} = -3a$ (stable), 0 (neutral) which matches $\bar{x} = 0, a$