Phys 325 Discussion 10 – Drilling Euler-Lagrange with Geodesics

Summary of Variational Calculus: If you want to extremize a quantity S that is integrated over some path $\{q_i(t)\}$ of your system between fixed endpoints, and S is described by the integral

$$\boxed{S = \int L(q_i(t), \dot{q}_i(t), t)\, dt} \quad \text{with fixed endpoints } t_1, q_i(t_1), \text{ and } t_2, q_i(t_2),$$

then the path $\{q_i(t)\}$ that extremizes this integral satisfies the **Euler-Lagrange (E-L) equation**

$$\boxed{\frac{\partial L}{\partial q_i} = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right)} \quad \text{for each coordinate } q_i$$

That's it! Solving calculus-of-variations problems is a 3-step game:

1. Set up the path integral S you are trying to extremize. Use "The Procedure" from 225 to set up path integrals, and remember that the endpoints must be fixed.
2. Play "match the letters" → What's the independent variable "t"?
   What are the dependent coordinates"q_i"? What's the Lagrangian "L"?
   Compare your path integral to the form in the first box above to figure it out.
3. Write down the E-L equation for each q_i, then solve them for the path $\{q_i(t)\}$ that extremizes S.

Drilling with Geodesics
Today, we will drill! drill! drill! on this common class of problems, using a few simplifications since this is new.

- All problems have only *one* dependent coordinate q, and so only one E-L equation.
- Each problem is a **geodesic** problem. A geodesic is a path that minimizes the distance between two points. This is exactly the sort of problem for which variational calculus is designed! Your task will be to find the path of shortest distance between two given points by extremizing the integral $\int dl$ in various situations.
- We won't spend any time *solving* the E-L equations (that's just math). Instead, each problem will be a familiar physical situation; you'll be able to intuit the solution in advance, then verify that it solves the E-L equation.

DOT-NOTATION NOTE: It is common to use **dot notation** to denote the derivative $\dot{q}_i \equiv dq_i / dt$ even if your independent parameter is **not time**. For example, if your independent variable is x and the path you seek is $y(x)$, you might find it strange to write the derivative dy/dx as $\dot{y}$ instead of y'. You may use y' if you like, but the checkpoints all use $\dot{y}$. The point is: there is no possible confusion between y' and $\dot{y}$ since y is only a function of one variable = the one independent variable that serves as the integration parameter for your system's path.

Problem 1 : Geodesic in the XY-plane with Cartesian coordinates *Checkpoints* [1]

We did this in lecture. Go straight to Problem 2 if you attend lecture.

(a) Find the E-L equation for the geodesic path $y(x)$ between the points $(x,y) = (0,0)$ and (x_0,y_0).
Remember, for all of these problems the steps are: 1. set up the integral, 2. play "match-the-letters", 3. write down the E-L equation for the path you seek. Also remember that we will not be obtaining the general solution of these equations, so do not perform the d/dt integral on the right-hand side of the E-L equ, $\dfrac{\partial L}{\partial q} = \dfrac{d}{dt}\left(\dfrac{\partial L}{\partial \dot{q}}\right)$, until you have a specific solution to test. (It will waste a lot of time needlessly.)

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[1] You don't attend lecture? You're missing out, friend! This class asks *awesome* questions that lead to *many* illuminating discussions.
(a) The independent coordinate is x, the dependent coordinate is y, and the Lagrangian is $L = \sqrt{1+\dot{y}^2}$

→ E-L equation for $y(x)$ is $0 = \dfrac{d}{dx}\left(\dfrac{\dot{y}}{\sqrt{1+\dot{y}^2}}\right)$, i.e. $\dfrac{\dot{y}}{\sqrt{1+\dot{y}^2}} = C$ (constant). (b) Self-checking

(b) Show that the straight-line path $\{\ x:0 \to x_0,\ \ y(x) = x\, y_0/x_0\ \}$ solves your E-L equation and also satisfies the given boundary conditions (i.e. matches the given endpoints).

Problem 2 : Geodesic in the XY-plane with 2D-Polar coordinates *Hints & Checkpoints* [2]

(a) Find the E-L equation for the geodesic path $r(\phi)$ between generic endpoints (ϕ_1, r_1) & (ϕ_2, r_2) in the *xy*-plane. Follow the steps: 1. set up the integral you want to extremize 2. play "match-the-letters" 3. write down the E-L equation for the path you seek. Step 1 is often the tough one; there's a hint in the checkpoint.

• **Strategy**: Do *not* perform the "d/dt" integral on the right-hand side of the E-L equation! That derivative is usually a huge mess: it is a *full derivative*, not a partial one, so it hits *every* term that has *any* dependence your independent coordinate t … which includes all your dependent coordinates and their derivatives. Taking "d/dt" is to be avoided unless absolutely necessary. We're about to get a candidate solution to test, so wait until then!

(b) Consider the specific endpoints $\phi_1 = 0,\ r_1 = X_0$ and $\phi_2 = \pi/2,\ r_2 = \infty$. Show that the path

$\left\{\ \phi : 0 \to \dfrac{\pi}{2},\ \ r(\phi) = \dfrac{X_0}{\cos\phi}\ \right\}$ solves your E-L equation and matches the given endpoints.

• **Strategy**: Stick with sin and cos! $\dfrac{d}{d\phi}\left(\dfrac{1}{\cos\phi}\right) = \left(\dfrac{\sin\phi}{\cos^2\phi}\right)$ … leave that as it is, do *not* introduce tangent, secant, or cosecant or you will *miss obvious trig identities* when they appear. In general, sticking with sin and cos while you're still manipulating expressions is good advice!

(c) What path *is* this? Use a sketch or a transformation to Cartesian coordinates to figure it out.

(d) This time, let's find the geodesic between two endpoints at the same angle, $\phi_1 = \phi_2 \equiv \phi_0$, but different radii, r_1 and r_2. Draw the situation, then write down a parametrization in polar coordinates for the path you *know* is the shortest distance between those two points.

(e) As you can see from the path you just parametrized, solving for $r(\phi)$ is *not* the way to proceed for these endpoints: the correct geodesic has r varying and ϕ fixed, and there is no way to express *that* in the form $r(\phi)$! Instead we should solve for $\phi(r)$, with r as our independent variable. Find the E-L equation for the geodesic path $\phi(r)$ in polar coordinates, then check that your answer to (d) does indeed solve it.

Problem 3 : Geodesic on a Cylindrical Surface *Checkpoints* [3]

(a) A cylindrical surface is centered on the *z*-axis and has radius b. Find the E-L equation for the geodesic path $z(\phi)$ on the surface of this cylinder between generic endpoints (z_1, ϕ_1) and (z_2, ϕ_2) .

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[2] (a) Hint: You're minimizing path length so $S = \int dl$. Next you need dl in 2D-polar coordinates. Write down the **line element** $d\vec{l}$ for (r,ϕ)-space, then take its magnitude to get dl. If you're stuck on $d\vec{l}$ itself, recall the 225 mantra: the line element is ***how far you move*** *when you* ***increase each of your coordinates*** *by a* ***little bit d<coordinate>***. If you're still stuck, (r,ϕ)-space is identical to the (s,ϕ)-subspace of cylindrical coordinates, and that's on your 3Dmath formula sheet.

Answer: $L = \sqrt{\dot{r}^2 + r^2}\ \to$ E-L is $\dfrac{r}{\sqrt{\dot{r}^2 + r^2}} = \dfrac{d}{d\phi}\left(\dfrac{\dot{r}}{\sqrt{\dot{r}^2 + r^2}}\right)$ (c) straight line from $(x,y) = (X_0,0)$ to (X_0,∞)

(d) straight radial line: $\left\{\begin{array}{l} r\ \ :\ r_1 \to r_2 \\ \phi(r) = \ \phi_0 \end{array}\right\}$ (e) $L = \sqrt{1 + r^2\dot{\phi}^2} \to$ E-L is $0 = \dfrac{d}{dr}\left(\dfrac{r^2\dot{\phi}}{\sqrt{1 + r^2\dot{\phi}^2}}\right)$ i.e. $\dfrac{r^2\dot{\phi}}{\sqrt{1 + r^2\dot{\phi}^2}} = C$ (constant)

[3] (a) $L = \sqrt{b^2 + \dot{z}^2}\ \to$ E-L is $0 = \dfrac{d}{d\phi}\left(\dfrac{\dot{z}}{\sqrt{b^2 + \dot{z}^2}}\right)$, i.e. $\dfrac{\dot{z}}{\sqrt{b^2 + \dot{z}^2}} = C$ (constant) (c) $z(\phi) = \dfrac{h\phi}{\pi}$ = half-turn spiral

(b) Show that $\dot{z}(\phi)$ = constant satisfies your E-L equation.

(c) $\dot{z}(\phi)$ = constant is the general solution for this geodesic, cool. Write down the specific solution for these endpoints: $(z_1,\phi_1)=(0,0)$ and $(z_2,\phi_2)=(h,\pi)$, and describe / sketch the shape of the path you obtain.

Problem 4 : Geodesic on a Spherical Surface *Checkpoints* [4]

(a) A spherical surface is centered on the origin and has radius R. The Earth's surface is a perfect example to guide your intuition! Find the E-L equation for the geodesic path $\theta(\phi)$ on this surface between generic endpoints (θ_1,ϕ_1) and (θ_2,ϕ_2).

(b) Consider the specific case of endpoints with the same polar angle, $\theta_1=\theta_2\equiv\theta_0$, but different azimuthal angles ϕ_1 and ϕ_2. In the Earth analogy, these would be points at the same latitude. What is the path of shortest distance between two such points? Try the solution $\theta(\phi)$=constant and show that it *only* works for a <u>specific endpoint-angle θ_0</u>. What is this θ_0? Sketch the situation and show that your finding makes perfect sense. ☺

(c) Now flip the problem around and obtain the E-L equation for the geodesic $\phi(\theta)$, i.e. with θ as the independent variable this time.

(d) Consider the specific case of endpoints with the same azimuthal angle, $\phi_1=\phi_2\equiv\phi_0$, but different polar angles θ_1 and θ_2. In the Earth's-surface analogy, these would be points at the same longitude. What is the path of shortest distance between two such points? You know what it is … write down the solution you *know* must be true, then check that it satisfies your Euler-Lagrange equation from part (c).

GENERAL STRATEGY ISSUE — What's the best "*t*"?
In almost all variational problems, you are free to choose your independent variable "*t*" from among the set of coordinates specifying your system's configuration. The best choice depends on the problem. In Problem 2(d,e) you found a case where one choice ($t=\phi$) was unable to handle a special class of solutions (those with ϕ fixed and r varying independently of ϕ), so you had to switch to the other choice ($t=r$). In this week's homework, you will find other considerations, e.g. picking an independent variable that produces a Lagrangian with a <u>cyclic coordinate</u> is often of enormous help. In any case:

If you run into trouble with one choice of "*t*", try another choice.

The best way to gain strategy-boosting experience is to solve variational problems with *different* choices of "*t*" and see how your choices affect the solution.

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[4] (a) $L=R\sqrt{\sin^2\theta+\dot{\theta}^2}\ \rightarrow$ E-L is $\dfrac{\sin\theta\cos\theta}{\sqrt{\sin^2\theta+\dot{\theta}^2}}=\dfrac{d}{d\phi}\left(\dfrac{\dot{\theta}}{\sin^2\theta+\dot{\theta}^2}\right)$

(b) E-L equation for $\theta(\phi)$=constant give $\cos\theta=0$, so this solution only works when the endpoints $\theta_1=\theta_2\equiv\theta_0$ — which the geodesic path *must* pass through — satisfy $\cos\theta_0=0$. The solution $\theta(\phi)$=constant thus only works when $\theta_0=90°$, i.e. for points on the **equator**. This fact is familiar if you've ever taken an airplane flight between cities at similar latitudes, e.g. New York to Paris. The airplane does *not* fly at constant latitude over the atlantic, its trajectory goes "up" toward the north pole then "back down" again.

(c) $L=R\sqrt{1+\dot{\phi}^2\sin^2\theta}\ \rightarrow$ E-L is $\dfrac{\dot{\phi}\sin^2\theta}{\sqrt{1+\dot{\phi}^2\sin^2\theta}}=C$ (constant) (d) Path = $\left\{\begin{matrix}\theta\ :\ \theta_1\rightarrow\theta_2\\ \phi(\theta)=\ \phi_0\end{matrix}\right\}$ = meridian of constant longitude

This path is a segment of a "**great circle**" = any circle on the surface of the earth whose center coincides with the Earth's center. Segments of great circles are the general solution for geodesics on a spherical surface. This explains e.g. why airplanes go over / near the north pole when travelling between cities in the northern hemisphere. Unfortunately, the parametrization of a great circle looks *spectacularly* ugly unless your endpoints lie on a meridian of constant ϕ (part (d)) or on the equator (part (b)).