Phys 325 Discussion 11 – Welcome to Lagrangian Mechanics

Procedure for Lagrangian Mechanics: In last week's lectures, we presented the elements of Lagrangian mechanics and worked some examples. This week we will *prove* that the approach is valid, but the proof will be much more meaningful to you if you have worked with the procedure first. Below, we will go through each step of the Lagrangian procedure using a simple example. This week's homework also presents these steps, so if you've started the homework already, you don't need to read the paragraphs describing each step.

Problem 1: Step-by-Step — Bead on a Fixed Ring *Hints & Checkpoints* [1]

The Example Problem: A bead of mass m is threaded on a circular wire ring of radius R. The ring is placed in the xy-plane, centered on the origin, and not allowed to move. Uniform gravity g points in the $-y$ direction, and there is no friction between the bead and the wire ring. Our goal is to determine the motion of the bead.

Step 1: Figure out the **number of degrees of freedom (DOF)** of the system. We will call it $\boldsymbol{n}$. It is the number of coordinates of the system that can be varied independently. To illustrate, a system composed of N point-like particles has $3N$ degrees of freedom, since each particle can move in three independent spatial directions. However, if the particles are subject to **constraint forces** (normal forces, tensions, the binding forces that keep the $\approx 10^{23}$ atoms of a rigid object in a fixed shape, etc), restrictions are imposed on how the particles can move. This reduces the number of degrees of freedom to $n = 3N - n_c$, where n_c is the number of **constraint equations** imposing restrictions on the coordinates' values. In summary, n is the minimum number of coordinates you need to completely describe the system's configuration, given its constraints.

(a) How many degrees of freedom, n, does our example problem have?

Step 2: Select the $\boldsymbol{n}$ **generalized coordinates** $\{q_1, q_2, \ldots q_n\}$ that you will use to describe the state of your system. These q_i can be positions, angles, combinations thereof, etc. They can be anything as long as they

- completely describe the system's configuration at any given moment, and
- are independent of each other, meaning that you can change one without changing any of the others.

(b) What generalized coordinate(s) could you choose for this problem?

Step 3: Calculate the **Lagrangian** $\boxed{L(q_i, \dot{q}_i, t) = T - U}$ of the system. T and U are the system's total kinetic and potential energy, written entirely in terms of your chosen generalized coordinates q_i and/or the independent variable t = time. U is the potential energy under the influence of all forces that can do any work; we call these **supplied forces**. (If there are any forces that *can* do work but *can't* be described by a potential – friction is the prime example – the Lagrangian prescription cannot be used without significant hacking.) The **constraint forces** cannot do any work by definition, so they do not appear in the Lagrangian at all. As we will prove, the Lagrangian prescription *itself* takes care of the constraint forces, which is one of its great virtues!

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[1] (a) 1 (b) If you are picking a coordinate from one of our 3 standard systems, the azimuthal angle ϕ is by far the best choice. The bead's x, y, or s coordinates may seem like reasonable alternatives, but they are a bit dangerous as they don't *completely* describe the bead's position : knowing the bead's x-coordinate, for example, only restricts it to one of *two* positions on the ring. If you are only interested in the motion of the bead on one half of the ring (e.g. if you only care about its motion near equilibrium), then one of these alternatives would be fine, but not very pleasant to work with. (c) $L = \frac{1}{2} m R^2 \dot{\phi}^2 - mgR\sin\phi$ (d) $R\ddot{\phi} = -g\cos\phi$

(e) Reminder 1: You'll need to make a Taylor approximation for small angles, as usual, but ϕ is *not* small in this situation; rather its *deviations* from ϕ_0 are small. Reminder 2: Change variables! Rewrite your equation of motion in terms of the variable $\varepsilon \equiv \phi - \phi_0$, which *is* small near equilibrium. Reminder 3: For a near-equilibrium analysis, $\varepsilon, \dot{\varepsilon}$, and $\ddot{\varepsilon}$ can all be made arbitrarily small; find the lowest order of $\varepsilon, \dot{\varepsilon}$, and/or $\ddot{\varepsilon}$ that appears in your EOM — that's the "lowest non-vanishing order" — then drop all terms of higher order. The final result is $\omega = \sqrt{g/r}$. (f) $L = \frac{1}{2} m\left(\dot{x}^2 + \dot{y}^2\right) - mgy$, constraint equation is $x^2 + y^2 = R^2$ (g) EOM: $\ddot{x} = 0$, $\ddot{y} = -g$

Important: The Lagrangian must be expressed in the form $L(q_i, \dot{q}_i, t)$: it must be written entirely in terms of your chosen generalized coordinates, their derivatives, time, and/or constants. *No other variables may appear!* The upcoming proof of the formalism will make it clear why this is necessary; below, we'll also test this restriction and explicitly *show* that it is necessary.

(c) Write down the Lagrangian for the bead in terms of $\phi, \dot{\phi}, t,$ and/or constants.
Suggestion: The generalized coordinates you choose are of paramount importance, so it's good practice to circle them right at the start of your work. Do it: write down ϕ and circle it! As you will discover, you usually have to introduce other variables when calculating your Lagrangian, so it is easy to lose track of your choice of q_i's.

Step 4: Apply the **Euler-Lagrange equations** to get the system's ***n* equations of motion**:

$$\boxed{\frac{\partial L}{\partial q_i} = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right)} \quad \text{for each generalized coordinate } q_i$$

In the context of mechanics, these are just called the **Lagrange equations**. (Prof. Euler is thus relegated to the realm of general variational calculus, i.e. pure mathematics.)

(d) Write down the bead's one equation of motion.

Step 5: **Solve** the equations of motion to determine the **system's behavior** $q_1(t), q_2(t), ..., q_n(t)$.

(e) The differential equation you obtained cannot be solved analytically for $\phi(t)$. It can be solved it in an approximate case, however: find the frequency ω of small oscillations of the bead around the stable-equilibrium position $\phi_0 = -\pi/2$. If you have forgotten the procedure for small-oscillation analysis, three reminders are provided in the checkpoint.

(f) That's how Lagrangian mechanics works! You go through the same 5 steps every time. Now let's address a common question: why did we make such a big deal in the first two steps about figuring out the number of DOF and choosing *exactly n* generalized coordinates? Answer: the entire formalism depends on it. You will see this when we go through the proof, but nothing builds intuition like experimentation! Let's try *violating* this rule and see what happens. ☺ Let's use x and y as our coordinates. We know that they're not independent of each other because of the ring, but why not apply that constraint at the *end* of the procedure instead of at the start? Let's try it!

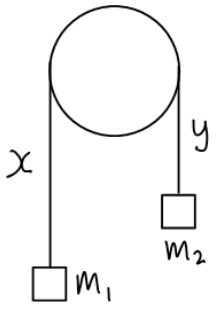
➔ Forget about the ring and write down the Lagrangian for the bead in terms of $x, \dot{x}, y, \dot{y}, t,$ and/or constants. It is a really easy task … this rebellious plan is going well so far! ☺

➔ Write down the constraint equation that relates x and y to each other because of the ring's shape. Our rebellious plan is to apply this constraint at the end, after we've determined the equations of motion.

(g) Since you have two coordinates, you will get two Euler-Lagrange equations of motion. Write them down.

(h) Now apply the constraint equation to replace all the y's with x's … and you will find that it's impossible. *The equations of motion are already hopelessly wrong*. That, friends, is the reason for the first two steps. ☺

Problem 2 : Atwood's Machine *Checkpoints* [2]



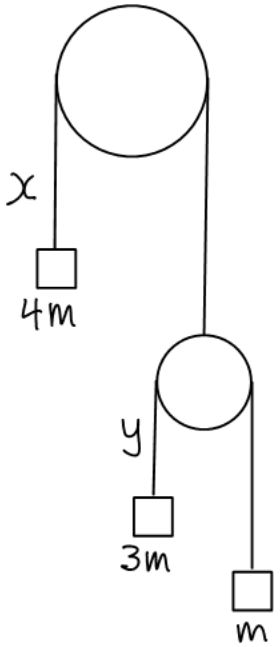
You remember Atwood's Machine from Homework 6? It consists of two masses tied to the ends of a massless string of length l, with the string running over a massless pulley whose axle is fixed in place.

(a) Use the Lagrangian prescription to find the accelerations $\ddot{x}$ and $\ddot{y}$ of m_1 and m_2.

(b) Was that easier than analyzing this device using Newton's force-based procedure? You decide! Remember, you have *tensions* and *normal forces* — constraint forces — to deal with in a force analysis, so you must draw free-body diagrams showing all the forces acting on each object (the two masses and the pulley, though for this particular problem you only need to analyze the forces m_1 and m_2). If you would like to get a feeling for Lagrange-vs-Newton, use forces to calculate $\ddot{x}$. The 3-in-1 free-body diagram is on the next page to help you.

(c) We already have an energy-based way to solve a system: **conservation of energy**. The Atwood machine obviously conserves $T+U$, and that gives us one EOM: $T+U$ = constant (1st-order form) or $\dot{T}+\dot{U}=0$ (2nd-order form). Since the Atwood machine has only one degree of freedom, *one EOM is all you need!* In part (a), you already calculated T and U and applied the string-length constraint to write those quantities in terms on one DOF (x or y). Calculate $\dot{T}+\dot{U}=0$ and see if gets you to the solution any faster than $\partial L/\partial x = d(\partial L/\partial \dot{x})/dt$.

Problem 3 : Double Atwood Machine *Checkpoints* [3]



The Lagrangian method didn't particularly speed up our analysis of the Atwood machine, but even for such a simple problem you can already see one of its advantages: by getting rid of all constraint forces, it requires *less thought* (if not less paper) and so fewer possibilities for errors. You also found that conservation of $T+U$ has the same advantages. Where the Lagrangian method really shines is for: (i) constrained systems with *more than one DOF*: more than one EOM is needed here, so even if $T+U$=constant, it's not enough to solve the problem (ii) systems where $T+U$ is *not* conserved. To see this in action, let's revisit the notorious double Atwood machine, with two massless pulleys and two massless strings. On Homework 6, your force-based analysis of this system required four separate EOMs to solve the system … let's see how Monsieur Lagrange does with this one! This system has two degrees of freedom; let's choose the x and y distances labelled on the figure as our generalized coordinates. Recall that the upper pulley has its axle fixed in place, but the lower pulley can move.

(a) Use the Lagrangian prescription to calculate the acceleration $\ddot{x}$ of the mass $4m$.
TECHNIQUE: Drop constant additive terms & constant scale factors from the Lagrangian. Reason: our only use for the Lagrangian is to plug it into the Lagrange equations and get our system's EOMs. These equations involve only the *derivatives* of L, so additive constants have no influence. The equations also have L *on both*

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[2] (a) Hint: This problem has only one DOF. You can pick x or y as your generalized coordinate, but *not both* (!) since they are related by the constraint equation $x + y = l$. Depending on your choice, the Lagrangian is $L(x,\dot{x},t) = \frac{1}{2}\dot{x}^2(m_1+m_2) + gx(m_1-m_2)$ or $L(y,\dot{y},t) = \frac{1}{2}\dot{y}^2(m_1+m_2) + gy(m_2-m_1)$. Final answers: $\ddot{x} = g(m_1-m_2)/(m_1+m_2)$ & $\ddot{y} = -\ddot{x}$ (b) self-checking

[3] (a) With constant terms and factors dropped, $L = 2\dot{x}^2 + \dot{y}^2 - \dot{x}\dot{y} + gy \rightarrow \ddot{x} = g/7$ and $\ddot{y} = 4g/7$ (b) you decide ☺

sides, so multiplicative constants cancel and have no influence. Example: if $L = 3m\ddot{x} - mgx + 4mg$, you can drop the constant term $4mg$ and the constant factor m and just use $L = 3\ddot{x} - gx$.

☞ Was that easier than the force-based analysis? For a reminder of its complexity, the 5-in-1 free body diagram you needed in Homework 6 is shown below.

Problem 4 : Cylinder on a Ramp; Generalized Momentum & Force *Checkpoints* [4]

A cylinder of mass m, radius R, and moment of inertia I (for rotation around its central axis) rolls without slipping straight down an inclined plane, which is at an angle α from the horizontal. Use as your generalized coordinate the cylinder's distance x measured down the plane from its starting point.

(a) Calculate the Lagrangian for the cylinder.

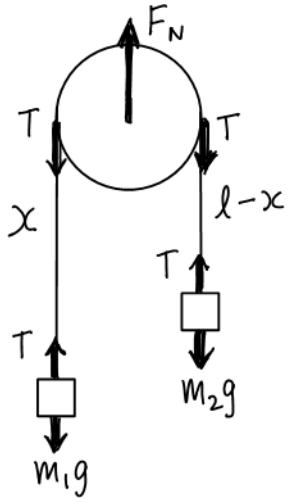
(b) We learned last week that the Lagrange equations can be written in the form

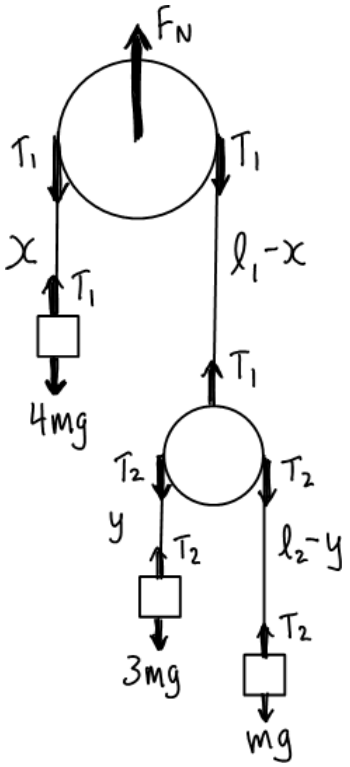
$\boxed{Q_i = \dfrac{dp_i}{dt}}$ where $Q_i \equiv \dfrac{\partial L}{\partial q_i}$ = **generalized force** and $p_i \equiv \dfrac{\partial L}{\partial \dot{q}_i}$ = **generalized momentum conjugate to q_i.**

What are Q_x and p_x for this system? As is typical, sometimes these quantities are recognizable components of force/torque/momentum/angular-momentum, and sometimes they are "effective" forces and momenta that you cannot readily identify with a familiar quantity.

(c) Use the Lagrangian procedure to solve the problem → determine the cylinder's acceleration $\ddot{x}$.

Free-body diagrams for single and double Atwood machines:





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[4] (a) $L = \dfrac{\dot{x}^2}{2}\left(m + \dfrac{I}{R^2}\right) + mgx\sin\alpha$ (b) $Q_x = mg\sin\alpha = F_x$, $p_x = \dot{x}\left(m + I/R^2\right)$ (c) $\ddot{x} = \dfrac{g\sin\alpha}{1 + I/mR^2}$