# Phys 325 Discussion 12 – Effective Potential and more Lagrangian Practice

Before Lagrangian mechanics, we had two methods for doing **equilibrium analysis** of a system:

- Method **A**: Analyze the first and second derivatives of the **potential energy** $U(q)$.
- Method **B**: Use the system's **EOM**(s) to find the equilibrium points, then **solve** the **Taylor-approximated** EOMs very near equilibrium in the style of a **small-oscillation analysis** to determine stability.

The potential-based Method A is fast but doesn't always work, while the EOM-based Method B always works but takes a bit longer. With Lagrangian mechanics, we still have the same two techniques, but we can extend Method A to more problems by introducing the notion of **effective potential**. Here's the idea: once you have built your Lagrangian, it no longer matters if its various terms came from $T$ or from $U$ … so if you have a term that *looks like* a potential in that it has no velocity-dependence, it will *behave* like a potential in the equations of motion. As we discussed in our last lecture, if the Lagrangian has 1 DOF and looks like this:

$$\boxed{L(q,\dot{q},t) = \frac{1}{2}\mu(q)\,\dot{q}^2 - U_{EFF}(q)} \quad \text{where}$$

- the **effective mass** $\mu(q)$ behaves like a normal mass: $\mu$ and $\mu'$ are finite at all possible equilibrium points
- the **effective potential** $U_{EFF}(q)$ behaves like a potential: it has no dependence on velocity or time

then an equilibrium analysis can be performed by applying Method A to the effective potential:

- The equilibrium points $\bar{q}$ are the points where $U'_{EFF}(\bar{q}) = 0$
- The equilibrium is stable at $\bar{q}$ if $U''_{EFF}(\bar{q}) > 0$
- The small-oscillation frequency at a stable-equilibrium point $\bar{q}$ is $\omega = \sqrt{U''_{EFF}(\bar{q})\,/\,\mu(\bar{q})}$

**Problem 1 : The proof** *Checkpoints* [1]

Very simple, just apply Method B to the Lagrangian form in the box. Here are the steps:

EOM: $\dfrac{\partial L}{\partial q} = \dfrac{d}{dt}\left(\dfrac{\partial L}{\partial \dot{q}}\right) \quad \rightarrow \quad -U'_{EFF} + \dfrac{1}{2}\mu'\,\dot{q}^2 = \mu\,\ddot{q} + \mu'\,\dot{q}^2$

Find equilibrium solutions $q(t) = \bar{q}$ constant: $\dot{q} = 0$ & $\ddot{q} = 0 \quad \rightarrow \quad -U'_{EFF}(\bar{q}) = 0$

Find EOM near equilibrium: shift coordinate $\varepsilon \equiv q - \bar{q} \quad \rightarrow$ EOM is $-U'_{EFF} - \dfrac{1}{2}\mu'\,\dot{\varepsilon}^2 = \mu\,\ddot{\varepsilon}$

Approximate EOM to lowest non-vanishing order in $\varepsilon$: $-\cancel{U'_{EFF}(\bar{q})} - \varepsilon\,U''_{EFF}(\bar{q}) - \dfrac{1}{2}\mu'\,\cancel{\dot{\varepsilon}^2} \approx \mu\,\ddot{\varepsilon}$

Solve approximate EOM: $\ddot{\varepsilon} \approx -\varepsilon\dfrac{U''_{EFF}(\bar{q})}{\mu(\bar{q})} \quad \rightarrow$ oscillates with $\omega = \sqrt{\dfrac{U''_{EFF}(\bar{q})}{\mu(\bar{q})}}$ if $U''_{EFF}(\bar{q}) > 0$

All you have to do is review or work through these steps, and make sure you understand everything. (This is a good review of Method B!) In particular, examine the second-last step: is it clear where $U''_{EFF}$ came from, and why the first and third terms were dropped? Ask your TA if you are unsure of anything at all.

Now let's work some equilibrium / small-oscillation problems and see if we can use effective potential to speed up the analysis. Note that you never *have* to use effective potential, Method B is always available. ☺

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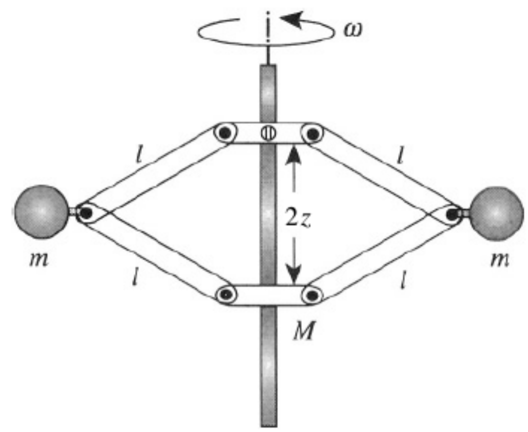
[1] In second-last step: • $U''_{EFF}$ came from Taylor expansion of $U'_{EFF}$ • 1st term dropped because it is the equilibrium condition • 3rd term dropped because it is second-order in ε while the other terms are first-order.

**Problem 2 : Flyball Governor** *Checkpoints* [2]




The apparatus shown in the figure is a centrifugal governor, invented in 1788 by James Watt to regulate the speed of a steam engine. The two "flyballs" of mass $m$ are connected by four hinged, massless rods of length $l$ to two sleeves wrapped around the vertical shaft. (The sleeves are the short horizontal bars in the figure.) The upper sleeve is fixed to the shaft, while the lower sleeve has mass $M$ and is free to move vertically. The shaft is connected to an engine that rotates it at constant angular velocity $\omega$.

(a) $z$ is the vertical distance between the flyballs and the sleeves (so the distance between the sleeves is $2z$, as shown in the figure). Calculate the system's Lagrangian using $z$ as your generalized coordinate. Use the symbol $\eta \equiv M / m$ to simplify your expression.

(b) Find the one equilibrium point $\bar{z} < l$.

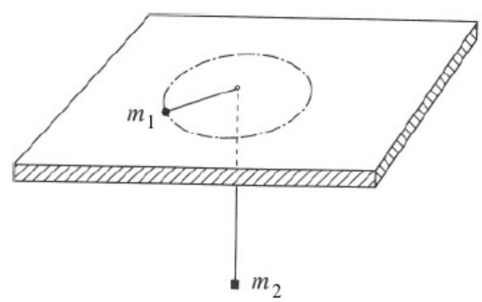
(c) Now set $\eta = 0$. (It doesn't change the problem, just reduces the algebra a bit to save you time.) Calculate the system's <u>small-oscillation frequency</u> around the equilibrium point you just found.

(d) Under what conditions is $\bar{z}$ a point of <u>stable equilibrium</u>? Note: you can figure that out immediately from the small-oscillation frequency you just found. If that is unclear to you, ask your TA!

(e) Does this system have any <u>constants of motion</u>? (i.e. conserved quantities). If so, find it / them; if not, explain why you are sure there are none. Continue to use the case $\eta = 0$ to simplify your analysis.

**Problem 3 : Table & String** *Checkpoints* [3]

Two <u>equal</u> masses $m_1 = m_2 = m$ are joined by a massless string of length d that passes through a hole in a frictionless horizontal table. The mass $m_1$ slides on the table while $m_2$ hangs below the table and moves up and down in a vertical line.




(a) Assuming the string remains taut, write down the Lagrangian for the system in terms of the polar coordinates $(s, \phi)$ of the mass on the table.

(b) Find all constants of motion of the system.

(c) Write down the two Lagrange equations of motion.

(d) As noted on your last homework, you can't make any progress toward a solution until you **separate** the EOMs, and constants of motion are very helpful for doing this! Use $s^2\dot{\phi} = l$ (constant) to express radial EOM entirely in terms of $s$, its derivatives, and constants. We have now reduced the problem to 1 DOF, $s$. Find the position $s_0$ where the radial coordinate $s$ is in equilibrium.

(e) Find the small-oscillation frequency of $s$ around $s_0$. (You can leave $s_0$ in your answer.)

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[2] (a) $L = \dot{z}^2\left(2\eta + \dfrac{l^2}{l^2 - z^2}\right) - z^2\omega^2 + 2(1+\eta)gz$ (b) $\bar{z} = g(1+\eta)/\omega^2$ (c) osc freq = $\omega\sqrt{1 - \dfrac{g^2}{\omega^4 l^2}}$ (d) stable if $g < \omega^2 l$

(e) one constant of motion: $H = \dot{z}^2 l^2 / (l^2 - z^2) + z^2\omega^2 - 2gz$

[3] (a) $L = \dot{s}^2 + \frac{1}{2}s^2\dot{\phi}^2 - gs$ (b) 2 constants of motion: $p_\phi = s^2\dot{\phi}$ and $H = T + U$. (c) $2\ddot{s} - s\dot{\phi}^2 = -g$ and $s^2\dot{\phi} = \text{const}$

(d) $s_0 = (l^2 / g)^{1/3}$ (e) $\omega = \sqrt{3l^2 / (2s_0^4)}$