Phys 325 Discussion 13 – Life in Accelerating Frames

This week we will study mechanics in non-inertial reference frames. A good way to think of a reference frame is as a coordinate system, i.e. an origin plus a set of $\hat{x},\hat{y},\hat{z}$ axes. A non-inertial frame is one that is **accelerating**: the origin may be accelerating linearly and/or the axes may be rotating. To picture this, let's consider an inertial frame S and an accelerating frame S*. Let's also bring back our favorite twins Alice and Bob from special relativity. Bob is always the stable guy, so we will put him in the inertial frame S. Alice is the more adventurous one, so we will put her in the S* frame. When the twins measure a location (e.g. the position of some moving object), Bob will use coordinates x, y, z and Alice will use coordinates x^*, y^*, z^*. What we will find is that Alice cannot use Newton's second law to analyze her world → the "F" in "$F = ma$", has to be modified in an accelerating frame to include **pseudo-forces**. The result is

$$m\vec{a}^* = \vec{F} + \vec{F}^*_{\text{pseudo}} \qquad \text{(* denotes measurement in accelerating frame)}$$

with four possible pseudo-forces:

$$\vec{F}^*_{\text{lin}} = -m\vec{A}_0\,, \qquad \vec{F}^*_{\text{cf}} = m\left(\vec{\Omega}\times\vec{r}^*\right)\times\vec{\Omega}\,, \qquad \vec{F}^*_{\text{Cor}} = m\,2\vec{v}^*\times\vec{\Omega}\,, \qquad \vec{F}^*_{\text{azim}} = m\vec{r}^*\times\dot{\vec{\Omega}}$$

Here $\vec{A}$ is the linear acceleration of the origin of S* and $\vec{\Omega}$ is the rotation vector for the axes of S* (both measured in the inertial frame). Let's consider the two cases separately.

Problem 1: Linear Pseudo-Force *Checkpoints* [1]

In this problem, Alice's coordinate system S* is not rotating but its origin is undergoing linear acceleration $\vec{A}_0$. You can imagine Alice to be in an accelerating railway car while Bob remains stationary at the train station. To make things simple, let's take $\vec{A}_0$ to be in the $+x$ direction.

(a) Make a quick sketch: draw one pair of axes $\hat{x},\hat{y}$ for Bob and, to the right of that, draw another pair $\hat{x}^*,\hat{y}^*$ for Alice, with both x axes pointing to the right. Indicate Alice's acceleration with a right-pointing vector $\vec{A}_0$ attached to the origin of her coordinate system. Now place an object of interest somewhere on your sketch with a dot. If Bob measures (x, y, z) for the position of this object, what position (x^*, y^*, z^*) does Alice measure? Express each of Alice's coordinates x^*, y^*, z^* as a function of Bob's coordinates x, y, z, the measurement time t, and the acceleration A_0. (Assume that Alice and Bob's coordinate systems coincide at time $t = 0$.)

(b) Using the equations you just wrote down, determine the acceleration $\vec{a}^* = (\ddot{x}^*,\ddot{y}^*,\ddot{z}^*)$ of the object as measured by Alice in terms of the acceleration $\vec{a} = (\ddot{x},\ddot{y},\ddot{z})$ measured by Bob.

(c) Alice measures $\vec{a}^* = (\ddot{x}^*,\ddot{y}^*,\ddot{z}^*)$ and tries to do a force calculation with Newton's second law. We know that $\vec{F} = m\vec{a}$ holds just fine in Bob's frame … but when Alice measures $m\vec{a}^*$, she does not get back the total physical force $\vec{F}$ on the object. What does Alice find for $m\vec{a}^*$? Express your answer in terms of $\vec{F}$, m, and the acceleration $\vec{A}_0 = A_0\hat{x}$ of Alice's frame.

We have our first pseudo-force! In a frame S* whose origin has acceleration $\vec{A}_0$, Newton's second law becomes

$$m\vec{a}^* = \vec{F} + \vec{F}^*_{\text{lin}} \qquad \text{with } \underline{\text{linear pseudo-force}} \quad \vec{F}^*_{\text{lin}} = -m\vec{A}_0$$

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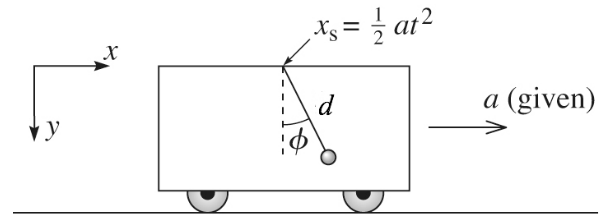
1 (a) $x^* = x - \frac{1}{2}A_0t^2,\ y^* = y,\ z^* = z$ (b) $(\ddot{x}^*, \ddot{y}^*, \ddot{z}^*) = (\ddot{x} - A_0, \ddot{y}, \ddot{z})$ (c) $m\vec{a}^* = \vec{F} - m\vec{A}_0$ (d) $\tan(\phi_0) = -a/g$

(e) $\omega^2 = \left|g^*\right|/d = \left|g\hat{y} - a\hat{x}\right|/d = \sqrt{a^2 + g^2}\,/\,d$

In words, Alice's world inside the railway car seems to have an additional force acting on everything: a force of magnitude mA_0 that points to the left. Since this pseudo-force is proportional to the mass m of every object, it behaves very much like gravity $\rightarrow$ the linear pseudo-force acts exactly like an additional source of uniform gravity with $-\vec{A}_0$ replacing $\vec{g}$. Let's see this at work with an example.

(d) You recently analyzed a pendulum inside an accelerating railway car, with acceleration $\vec{A}_0 = a\,\hat{x}$. You made no use of pseudo-forces at all since you analyzed everything in the inertial frame of the ground. Using whatever technique you want (force analysis or Lagrangian) determine the pendulum's equilibrium angle ϕ_0 in terms of a and g.



(e) On your homework, you also calculated the small-oscillation frequency of the pendulum using the Lagrangian technique. It was not the same as the usual frequency $\omega^2 = g/d$ for a pendulum in uniform gravity g. A significant amount of work was required to come up with the Lagrangian $L(\phi,\dot{\phi},t) = m(\frac{1}{2}d^2\dot{\phi}^2 + a\,d\,t\,\dot{\phi}\cos\phi + \frac{1}{2}a^2t^2 + gd\cos\phi)$, which is pretty ugly, then more work to obtain the oscillation frequency. Let's try out our new technique on this problem! Redraw the railway car as seen by Alice, who is *in* the car. She does *not* realize that she is accelerating, but she has measured her world and mapped out the forces acting on massive objects. As we discussed above, she finds that the masses in her world experience both actual gravity and a linear pseudo-force $\rightarrow$ add those together, and you get an effective total gravitational acceleration that you can call $\vec{g}^*$. If you think about the pendulum from Alice's perspective, you will quickly obtain the pendulum's oscillation frequency. What is it?

Switching sides: Newton's Law "$F = ma$" really is working in both frames, we just have to include pseudo-forces on the "F" side if we are working in an accelerating frame. Here's what has really happened: the railcar's acceleration is on the "a" side in Bob's frame (where it is part of the pendulum's acceleration) but moves over to the "F" side in Alice's frame. That is why pseudo-forces are called pseudo-forces: they do not originate from any physical force like gravity but purely from the kinematics of an accelerating frame.

Problem 2: Alice, Bob, and a Turntable *Checkpoints* [2]

Now let's turn to the case of a rotating reference frame. This time, the axes of Alice's coordinate system are rotating, with angular velocity vector $\vec{\Omega}$. Alice will find three pseudo-forces acting on an object of mass m, position $\vec{r}^*$, and velocity $\vec{v}^*$; they are the **centrifugal**, **Coriolis**, and **azimuthal** pseudo-forces, and their formulae are on the next page.

To get a handle on rotating frames, think about a flat turntable. We will call $\hat{z}$ the "upward" direction, have the table parallel to the ground, and set it rotating with angular speed $\vec{\Omega} = \omega\,\hat{z}$. Bob and Alice both chose the center of the turntable as their origin, but Bob's coordinate system is stationary with respect to the ground while Alice's is anchored to the turntable. As usual, Alice is unaware of her environment, so imagine that the turntable is surrounded by a cylindrical wall half-silvered glass that prevents Alice from looking outward, but allows Bob to look inward and see what Alice is doing. To complete their coordinate systems, Alice and Bob each place a red dot somewhere to indicate the $+x$ direction: Bob paints his red dot somewhere on the ground, and Alice paints her red dot somewhere on the wall of her cylindrical world. Bob and Alice each use their

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2 (a) $s = R,\ \phi = 0,\ \vec{v} = 0,\ \vec{a} = 0$, no forces (b) $s^* = R,\ \phi^* = -\omega t,\ \vec{v}^* = -R\omega\,\hat{\phi},\ \vec{a}^* = -R\omega^2\hat{s},\ \vec{f}_{cf} = R\omega^2\hat{s},\ \vec{f}_{Cor} = -2R\omega^2\hat{s}$

(c) $s = v_0 t,\ \phi = 0,\ \vec{v} = v_0\hat{s},\ \vec{a} = 0$, no forces; $s^* = v_0 t,\ \phi^* = -\omega t,\ \vec{v}^* = v_0\left(\hat{s} - \omega t\hat{\phi}\right),\ \vec{a}^* = -v_0\omega\left(\omega t\,\hat{s} + 2\hat{\phi}\right)$,

$\vec{f}_{cf} = v_0\omega^2 t\,\hat{s},\ \vec{f}_{Cor} = -2v_0\omega\left(\omega t\,\hat{s} + \hat{\phi}\right)$ (d) There must be an external force this time! $\vec{f}_{EXT} = v_0\omega\left(-\omega t\,\vec{s} + 2\hat{\phi}\right)$

(f) Walking in the Southern hemisphere is analogous to walking on a turntable with $\vec{\Omega}$ upside down $\therefore$ you are pushed to the left.

personal Origin-to-RedDot line as their $+x$ axis and will measure azimuthal angles relative to that: ϕ for Bob and ϕ^* for Alice. Finally, here are the formulae for the three rotational pseudo-forces:

$$\text{Pseudo-Forces in rotating frame:} \qquad \vec{F}^*_{\text{cf}} = m\left(\vec{\Omega}\times\vec{r}^{\,*}\right)\times\vec{\Omega}\,, \qquad \vec{F}^*_{\text{Cor}} = m\,2\vec{v}^{\,*}\times\vec{\Omega}\,, \qquad \vec{F}^*_{\text{azim}} = m\vec{r}^{\,*}\times\dot{\vec{\Omega}}$$

In each of the following examples, Alice will execute a simple movement. To highlight the pseudo-forces, we will have **no real forces** acting on Alice. This is easily accomplished by giving Alice frictionless shoes and a jet pack that allows her to maneuver ☺. Gravity is irrelevant since we will confine all our motions to the plane of the turntable. (Nothing interesting happens in the z direction anyway, as you can see from the pseudo-force equations.) Your task for each example is to determine these four things:

➔ Alice's • coordinates s,ϕ • velocity vector • acceleration vector • all the forces acting on her,

each as a function of time … but of course you must do it *twice*: once in Bob's frame and once in Alice's frame. Then we will learn something! You will see that "***F = ma***" always holds, for everyone … it's just that observers have different opinions about what's an "F" and what's an "a". You may need these formulae:

$$\text{Cylindrical Coord:} \qquad \vec{v} = \dot{s}\,\hat{s} + s\dot{\phi}\,\hat{\phi} + \dot{z}\,\hat{z}\,, \qquad \vec{a} = \hat{s}\left[\ddot{s} - s\dot{\phi}^2\right] + \hat{\phi}\left[s\ddot{\phi} + 2\dot{s}\dot{\phi}\right] + \hat{z}\left[\ddot{z}\right]$$

We know how everything works in Bob's world, so we will determine how things look to Alice by making a coordinate change: with $\vec{\Omega} = \Omega\hat{z}$, the change from S-frame to S*-frame is { $z = z^*$, $s = s^*$, $\phi = \phi^* + \Omega t$ }.

(a) Bob sees Alice standing at rest a distance R from the origin. Find the requested quantities as determined by Bob: $s(t)$, $\phi(t)$, $\vec{v}(t)$, $\vec{a}(t)$, and all forces-per-unit-mass $\vec{f}(t) \equiv \vec{F}(t)/m$.

(b) Same situation: Alice is standing still in the inertial frame at a point R meters away from the origin … but what does this situation look like in Alice's frame? Find $s^*(t)$, $\phi^*(t)$, $\vec{v}^{\,*}(t)$, $\vec{a}^{\,*}(t)$, all $\vec{f}^{\,*}(t)$, and most importantly, SKETCH the trajectory.

(c) Next situation: Alice moves at constant speed v_0 along Bob's x-axis, starting from the origin at t=0. Analyze this motion in Bob's frame, then in Alice's frame, and SKETCH BOTH.

(d) Last situation: Alice moves at constant speed v_0 along her own x^*-axis, again starting from the origin at t=0. Analyze everything … and you will find something different about this case than (a)-(c).

(e) Convince yourself that if you move in *any* direction on the turntable, the Coriolis force you experience always "**pushes you to the right**". This is a common way of summarizing the effect of the Coriolis force, make sure you see what it means.

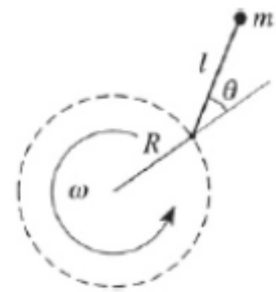
Though the Coriolis force is unintuitive at first, its origin is truly trivial, and is completely explained by the thought experiments of parts (c) and (e): when a particle moves in a straight line in Bob's frame, the rotating table moves underneath it … so in Alice's turntable frame, the particle's trajectory bends. Give a particle some initial velocity vector in a rotating frame and apply no real forces to it; since your *frame turns as the particle moves* along what is *actually* a straight line trajectory (as no real forces are acting)*, you see its trajectory as curved*. You conclude there is a force causing this curvature. That pseudo-force is Coriolis.

(f) Finally put yourself on the Earth's surface. The "Coriolis pushes you to the right" rule applies if you are in the Northern Hemisphere … but not in the Southern hemisphere. What happens there? Why?

There's one final problem on the next page, keep going. :-)

Problem 3: Pendulum Clock (Maryland qual exam) *Guidance* [3]

An off-duty physicist designs a pendulum clock for use on a gravity-free spacecraft. The mechanism is a simple pendulum (mass m at the end of a massless rod of length l) hung from a pivot, about which it can swing in a plane. To provide artificial graivty, the pivot is forced to rotate at a frequency ω in a circle of radius R in the same plane as the pendulum arm. Show that this succeeds, i.e. that the possible motions $\theta(t)$ of this pendulum are identical to the motions $\theta(t)$ of a simple pendulum in a uniform gravitational field of strength $g = \omega^2 R$, not just for small oscillations but for any amplitude, and for any length l, even $l > R$.



3 Guidance 1: Here is a great example where you may be tempted to try an analysis in an accelerating frame … but you will probably find it easier to work in an inertial frame. This is not always the case, it depends entirely on the problem, and an outstanding learning tool is to try both. Guidance 2: There is a very good way to avoid using / remembering / deriving the formulas for accelerating in non-Cartesian coordinate systems: avoid vectors entirely by using Lagrangians. ☺ Yes!