

Phys 325 Discussion 14 – Life Goes on in Accelerating Frames

Summary: Since all the pseudo-forces are proportional to mass, we first define $\vec{f} \equiv \vec{F} / m =$ force per unit mass. This quantity is actually familiar to you already, from gravity: the gravitational acceleration g (the familiar 10 m/s^2 at the Earth’s surface) is nothing more than $\vec{g} \equiv \vec{f}_{\text{grav}}$. The pseudo-forces-per-unit-mass, \vec{f}^* , play the exactly same role as \vec{g} , but they only appear in accelerating frames.

$$\text{Pseudo-forces / mass: } \vec{f}_{\text{lin}}^* = -\vec{A}_0, \quad \vec{f}_{\text{cf}}^* = (\vec{\Omega} \times \vec{r}^*) \times \vec{\Omega} = \Omega^2 s^* \hat{s}^*, \quad \vec{f}_{\text{Cor}}^* = 2\vec{v}^* \times \vec{\Omega}, \quad \vec{f}_{\text{azim}}^* = \vec{r}^* \times \dot{\vec{\Omega}}$$

$$\text{Cylindrical Coordinates: } \vec{v} = \dot{s} \hat{s} + s \dot{\phi} \hat{\phi} + \dot{z} \hat{z}, \quad \vec{a} = \hat{s} [\ddot{s} - s \dot{\phi}^2] + \hat{\phi} [s \ddot{\phi} + 2 \dot{s} \dot{\phi}] + \hat{z} [\ddot{z}]$$

Problem 1: Is Our Plumb Line Tilted or Not?

*Hints & Checkpoints*¹

A plumb line is a device used by carpenters to determine the true vertical direction. It is just a string with a mass attached at one end; you hold the other end in the air, and the string displays “vertical” for you. But what is “vertical”, exactly, in the rotating reference frame of the Earth?

(a) Use R as the earth’s radius, ω as the earth’s rate of rotation, and g as the acceleration due to gravity. Let the z axis run from the South pole to the North pole and place the origin at the earth’s center; we can now use standard coordinates like the polar angle θ to denote positions on the earth’s surface and standard unit vectors like \hat{r} and \hat{s} to describe directions. So: consider a carpenter who is standing at a location with polar angle θ and is using a plumb line. The plumb line is at rest in the carpenter’s frame (that’s how you use a plumb line!), and the carpenter’s frame is (of course) the rotating frame of the earth. Obtain an expression for the angle α between the plumb line and the line pointing from the carpenter to the earth’s center. Hint: the direction of the plumb line is the direction of $\vec{f}_{\text{total}} = \vec{f}_{\text{grav}} + \vec{f}^*$ on the plumb line’s mass. Further hints are in the checkpoint.

Also, the expression will be fairly ugly ... don’t worry about it. ☺

(b) Now that you have your fairly ugly expression for α , it’s time to approximate! Your result contains a combination of the gravitational and centrifugal forces. Well, we know from personal experience that the outward-pointing centrifugal force $\vec{f}_{\text{cf}}^* = \omega^2 R \sin \theta \hat{s}$ has to be a lot smaller than inward-pointing gravity

$\vec{f}_{\text{grav}}^* = -g \hat{r}$ or we would all fly off the earth’s surface! There is clearly a small quantity in here somewhere ...

Your task is to find a dimensionless quantity that is much less than 1. It will clearly be related to the relative sizes of the centrifugal and gravitational forces, so it will clearly be some combination of ω , R , and g . Use the values $R = 6.4 \times 10^6 \text{ m}$ and $g = 10 \text{ m/s}^2$; you can calculate the earth’s angular speed ω yourself, since you live here. ☺ Note that a finding of the form $a \ll b$ is exactly what you want: if a and b can be compared, they have the same units, so a/b must be dimensionless and much less than 1 \rightarrow exactly what you’re looking for.

¹ (a) Hint 1: The plumb line is at rest in the earth’s frame, so $v^* = 0$... that means only one of the pseudo-forces contributes.

Hint 2: Depending on how you proceed, you may find yourself needing to a dot-product of unit vectors like $\hat{s} \cdot \hat{r}$. How? Draw the Magic Triangle from 225: draw a random point in the z - s plane (a 2D sketch, not 3D), indicate the coordinates z, s, θ, r of the point on the sketch, then draw in the unit vectors you need. The sketch will give you whatever dot- or cross-production you want, in terms of

whatever coordinates you want, whee! ☺ Answer: either $\tan \alpha = \frac{\omega^2 R \sin \theta \cos \theta}{g - \omega^2 R \sin^2 \theta}$ or $\cos \alpha = \frac{g - \omega^2 R \sin^2 \theta}{\sqrt{g^2 - 2g\omega^2 R \sin^2 \theta + (\omega^2 R)^2 \sin^2 \theta}}$

(b) $\omega^2 R_{\oplus} / g \approx 3 \times 10^{-3} \ll 1$ (c) $\alpha \approx (R\omega^2 / g) \sin \theta \cos \theta$ (d) $\max \alpha \approx 0.1^\circ$ at $\theta=45^\circ$; $\alpha = 0$ at poles & equator (e) no! but close

- (c) Now take your fairly ugly expression from (a) and use your Taylor skills to find an approximate expression for the angle α that is accurate to leading non-vanishing order in the small quantity you found in (b).
 Hint 1: Your expression and your small-quantity should quickly reveal that the angle α is *also* small.
 Hint 2 = 225 Taylor-techniques: “*factor out the big term*” and “*dimensionless quantities are awesome*”.
- (d) Estimate the maximum and minimum values of the magnitude of α on the surface of the earth.
- (e) So: if you are using a plumb line here in Illinois, does its string point directly toward the center of the earth?

Problem 2: Spinning Bucket

Checkpoints ²

I am spinning a bucket of water about its axis of symmetry with angular velocity ω . (This axis runs down the center of the bucket; we are spinning the bucket around its own centerline, not twirling it around in a circle.) Show that, once the water has settled in equilibrium (relative to the bucket), its surface will be a parabola. Guidance: use cylindrical coordinates with the z axis running along the bucket’s axis of symmetry (i.e. parallel to $\vec{\omega}$), and remember that the surface is an equipotential under the combined effects of the gravitational and centrifugal forces.

Problem 3: Mass on a Rotating Plane

Checkpoints ³

A particle of mass m is confined to move, without friction, in a vertical plane (e.g. imagine that the mass m is sandwiched between two frictionless parallel plates.) The plane is forced to rotate with constant angular velocity ω about the $+y$ axis, which we will define as pointing upwards. Gravity g points in the downward direction. The S^* frame is the rotating frame of the plane, so let’s set up a “body-centered” coordinate system for our work in S^* → let $+y^*$ be upward (identical to $+y$) and let the x^* axis be horizontal (across the plane). Find the equations of motion for the x^* and y^* coordinates of the particle in its own frame (the rotating S^* frame), then solve those equations to find the general solution for $x^*(t)$ and $y^*(t)$ that describes all possible motions of the particle.

Problem 4: Plumb Line on a Train

Checkpoints ⁴

A high-speed train is traveling at a constant speed of 150 m/s (about 300 mph) on a straight, horizontal track across the South Pole. Find the angle between a plumb line suspended from the ceiling inside the train and another plumb line inside a hut on the ground. In what direction is the plumb line on the train deflected? (Find some way to describe the direction. ☺)

² Equation describing equipotential surface: $mgz - m\omega^2 s^2 / 2 = \text{constant}$, which gives $z = \omega^2 s^2 / 2g + \text{const} = \text{a parabola}$

³ General solution: $y^*(t) = y_0 + v_{y0}t + \frac{1}{2}gt^2$ and $x^*(t) = Ae^{\omega t} + Be^{-\omega t}$

⁴ Angle between plumb lines on train and in hut is $\arctan(2v\omega / g) = 0.13^\circ$; train’s plumb line hangs to the left of vertical if you are standing on the train and facing in the same direction as the train is moving.