Phys 325 Discussion 15 – Complex Numbers and Driven Oscillators

Problem 1 : Linear Differential Equations ➔ Superposition *Checkpoints*[1]

Linear differential equations (**LDEs**) are a very friendly class of differential equations that always merit special sections in textbooks. We must understand exactly what they are and why they are so friendly. ☺
To simplify this discussion, let's stick with one dependent and one independent variable, namely differential equations for a 1-dimensional function $x(t)$. The dependent variable is x and the independent variable is t.

- What exactly is a **linear** differential equation for $x(t)$?
  It is one where all terms are linear in the dependent variable x and/or its derivatives ($\dot{x}$, $\ddot{x}$, $\ldots$). Terms like $\dot{x}^2$, $\sqrt{x}$, $1/\ddot{x}$, e^x, $\ln(x)$, or $\ddot{x}x$ are forbidden as they involve x and/or its derivatives at powers other than 1.
  In contrast, the independent variable t may appear at any order.

- Can a **term without ANY appearance of *x*** be present in a linear differential equation?
  Yes. If there is no such term, we have a **homogeneous** linear equation; its most general form is:
  $$0 = a_0(t)x + a_1(t)\dot{x} + a_2(t)\ddot{x} + \cdots = a_n(t)x^{(n)}$$
  If our equation does have a term where no x appears (only t or constants), we have an **inhomogeneous** linear equation; its most general form is:
  $$f(t) = a_0(t)x + a_1(t)\dot{x} + a_2(t)\ddot{x} + \cdots = a_n(t)x^{(n)}$$
  In those compact forms on the right-hand-sides, we've used two conventions: (1) the Einstein convention that the repeated index n implies a sum over all n and (2) the use of a superscript in parentheses, (n), to denote an n^{th}-order derivative, e.g. $x^{(2)} \equiv \ddot{x}$.

- Why are linear differential equations "friendly"?
  One word: **superposition**. The purpose of this problem is to explore that supremely useful property!

(a) First let's consider homogeneous LDE's and use the most general form of such a thing for a function $x(t)$:
$$0 = a_0(t)x + a_1(t)\dot{x} + a_2(t)\ddot{x} + \cdots = a_n(t)x^{(n)}$$
Suppose we have found two solutions of this equation, $x_1(t)$ and $x_2(t)$.

> **Principle of Superposition for Homogeneous LDEs :** If $x_1(t)$ and $x_2(t)$ are solutions of a homogenous LDE, then any linear combination $x(t) \equiv C_1x_1(t) + C_2x_2(t)$ is also a solution.

Your task is to explicitly prove this extremely important principle so that it is forever cemented in your mind! Plug the proposed solution $x(t) \equiv C_1x_1(t) + C_2x_2(t)$ into the general homogeneous ODE above, use the fact that x_1 and x_2 are both solutions, and the proof will be done. To neaten your work, here is a helpful notational trick from our math friends: use the symbol D to denote the entire differential *operation* on the right-hand-side of the equation above. Here's what I mean: write your differential equation as
$$D[x] = 0 \qquad \text{with} \qquad D[x] \equiv a_0(t)x + a_1(t)\dot{x} + a_2(t)\ddot{x} + \cdots = a_n(t)x^{(n)}$$
The fact that x_1 and x_2 are both solutions is thus written $D[x_1] = 0$ and $D[x_2] = 0$. Your task is to show that $D[C_1x_1 + C_2x_2] = 0$ too, given the linear differential operator we are considering here.

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[1] (a)-(d) are self-checking, but if you need a hint, the key to (a),(c),(d) – the parts involving *linear* differential equations – is to demonstrate that $D[C_1x_1 + C_2x_2] = C_1D[x_1] + C_2D[x_2]$ for a linear differential operator $D[x] = a_n(t)x^{(n)}$. (e) Take f_1 and $f_2 = 0$ ☺

(b) To really cement a principle in your mind you need to explore a *counter-example* as well. Consider this differential equation: $0 = x + \dot{x}^2$. It is *not* linear. Again, suppose you have two solutions, $x_1(t)$ and $x_2(t)$. Show that the linear combination $x(t) \equiv C_1 x_1(t) + C_2 x_2(t)$ does <u>not</u> solve the <u>non-linear</u> equation. By working it out explicitly, you will see exactly why that non-linear $\dot{x}^2$ term destroys the superposition principle!

(c) Now consider <u>inhomogeneous</u> LDE's, again using the most general form for a function $x(t)$:

$$D[x] = f(t) \quad \text{where} \quad D[x] \equiv a_0(t)x + a_1(t)\dot{x} + a_2(t)\ddot{x} + \cdots = a_n(t)x^{(n)}$$

Given two solutions $x_1(t)$ and $x_2(t)$ of this equation, see if the linear combination $C_1 x_1(t) + C_2 x_2(t)$ also works.

(d) Hopefully you found that it does *not* work! Nevertheless, there is a more general superposition principle:

> **General Principle of Superposition :** Let $D[x]$ be a linear differential operator.
> If we have two functions $x_1(t)$ & $x_2(t)$ such that $D[x_1] = f_1(t)$ & $D[x_2] = f_2(t)$,
> then the linear combination $x(t) = C_1 x_1(t) + C_2 x_2(t)$ solves $D[x] = C_1 f_1(t) + C_2 f_2(t)$.

Proving that should be a one-line affair given all the work you've done above. ☺

Note: our previous superposition principle for *homogeneous* LDEs follows immediately from the general principle by simply taking $f_1 = f_2 = 0$.

Problem 2 : Differential Equations via Complex Numbers *Checkpoints* [2]

In lecture, we solved a particular LDE – the damped linear oscillator – by introducing complex numbers. Complex numbers allow us to treat exponentials and trig functions in the same way, and easily arrive at solutions where they are mixed together. You may be wondering: *when* exactly can we use this technique? To find out, we must first define what the technique *is* exactly. We'll do this with an example.

(a) First consider a <u>homogeneous</u> LDE, $D[x] = 0$, where D is a linear differential operator ($D[x] = a_n(t)x^{(n)}$).
Since D is linear, the crucial superposition property $D[C_1 x_1 + C_2 x_2] = C_1 D[x_1] + C_2 D[x_2]$ holds.
The **Switch-to-Complex Technique** goes like this:

> Goal: find the general solution $x(t)$ of the real equation $D[x] = 0$
> 1. Replace $x(t)$ with a complex function $\tilde{z}(t)$.
> 2. Solve the complex equation $D[\tilde{z}] = 0$ to obtain its general solution $\tilde{z}(t)$.
> 3. The general solution $x(t)$ is the real part of the complex solution: $x(t) = \text{Re}[\tilde{z}(t)]$.

<u>This procedure is always valid for a homogenous LDE</u>. To prove the procedure's validity, write $\tilde{z}(t)$ in the form $\tilde{z}(t) = x(t) + i\,y(t)$, then use superposition to show that if $D[\tilde{z}] = 0$, then $D[x] = 0$ as well.

(b) Counter-example! Apply the switch-to-complex trick to the *non-linear* equation $0 = x + \dot{x}^2$. Success?

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[2] (a) The key is that complex equations are two independent equations in one: $x_1 + i\,y_1 = x_2 + i\,y_2$ can only be true if $x_1 = x_2$ <u>and</u> $y_1 = y_2$. (b) The trick does *not* work for non-linear equations. Steps: change the equation to $0 = \tilde{z} + \dot{\tilde{z}}^2$, expand $\tilde{z}$ into its real and imaginary parts ($x + iy$), plug that in to $0 = \tilde{z} + \dot{\tilde{z}}^2$, and see if $x = \text{Re}[\tilde{z}]$ is a valid solution of the original equation $0 = x + \dot{x}^2$.
(c) yes (d) self-checking

(c) So this trick only works for linear differential equations. What about inhomogeneous linear equations? Say we want to solve $D[x] = f(t)$. We decide to go to complex numbers and solve $D[\tilde{z}] = f(t)$ first. If we find a complex solution $\tilde{z}(t)$, will $x(t) \equiv \text{Re}[\tilde{z}(t)]$ solve our original equation?

(d) Finally, make the inhomogeneous term $f(t)$ complex as well → change it to $\tilde{f}(t) \equiv f(t) + ig(t)$. Prove the following general result about the use of complex numbers to solve linear differential equations:

> Let D be a linear differential operator and define $\tilde{z}(t) \equiv x(t) + i\,y(t)$ & $\tilde{f}(t) \equiv f(t) + ig(t)$.
>
> If $\tilde{z}(t)$ solves $D[\tilde{z}(t)] = \tilde{f}(t)$, then $x(t)$ solves $D[x(t)] = f(t)$.

This is a powerful result → we can add any imaginary function we want to the inhomogeneous term $f(t)$, and the real part of the solution still solves the real part of the equation!

Friendly vs Helpful: Though the "switch-to-complex" technique always *works* for LDEs, it is not always *helpful*. In general, this technique is only useful when the solution is of the exponential form $e^{\tilde{\omega}t} = e^{\omega_R t} e^{i\omega_I t}$, i.e. an exponential whose exponent has a real part ω_R (which gives a true exponential) and/or an imaginary part ω_I (which gives a sinusoidal function). Damped oscillators are ideal candidates for complex numbers, as their solution combines *both* sinusoidal and exponential behaviour.

Problem 3 : Driven Oscillators

A child on a swing is an oscillator: a restoring force (gravity) always pulls the swing back to vertical.
A child on a rusty swing is a damped oscillator: the rust introduces a damping force that dissipates energy.
A child on a rusty swing that you push periodically is a driven, damped oscillator: you supply an additional force to maintain the oscillations. The general EOM for a linear, driven, damped oscillator is

$$\ddot{x} + 2\beta\dot{x} + \omega_0^2 x = f(t)$$

$f(t)$ on the right-hand side is the driving force; if it's zero, you have the damped oscillator we solved in class. To find the general solution, every textbook on every planet immediately does this:

☞ Restrict your attention to **sinusoidal driving forces** only: $f(t) = f_0 \cos(\omega t)$.

Why? Because Fourier's theorem allows you to build any $f(t)$ you like from a sum of such functions! Since the EOM is linear, superposition applies, so the solutions for driving frequencies ω_1, ω_2, ω_3, ... will add together nicely to give the solution for a linear combination of driving frequencies.

(a) Our goal is to find the general solution $x(t)$ for the equation $\ddot{x} + 2\beta\dot{x} + \omega_0^2 x = f_0 \cos(\omega t)$. Easy: rewrite the equation in complex form, guess the solution form, then plug it back in to make sure it works. The solution form you will guess is the usual exponential, $\tilde{x}(t) = \tilde{A} e^{\tilde{\Omega} t}$, with possibly-complex amplitude $\tilde{A}$ and possibly-complex exponent $\tilde{\Omega}$ … but first think physically: *how will this system behave*? If you apply a steady driving force of frequency ω to a spring whose natural undamped frequency is ω_0,

- will the spring oscillate at ω or at ω_0?
- will the spring's oscillations be damped or not?

(Think about the child you are pushing on the rusty swing.) Based on your thoughts, write down a more specific guess than $\tilde{x}(t) = \tilde{A} e^{\tilde{\Omega} t}$ → what is that exponent $\tilde{\Omega}$ going to be?

Check the checkpoint at the bottom before proceeding. *Checkpoints* [3]

(b) That was the most important step! By thinking physically, you restricted your solution form to $\tilde{x}(t) = \tilde{A}e^{i\omega t}$, where ω is the <u>driving frequency</u>. All we have left to determine is $\tilde{A}$. Go for it! Plug the solution form into our EOM to obtain the complex amplitude $\tilde{A}$ in the form $A e^{-i\delta}$ (i.e. obtain formulae for A and $\tan(\delta)$ in terms of the given parameters ω, ω_0, f_0, and β.)

(c) Have a look at your solution for $\tilde{x}(t)$: how many <u>free parameters</u> does it have?

(d) Your answer should reveal a problem. Do you recall from your differential equations class how to fix it? We are solving a 2nd order ODE … the general solution must have two free parameters, how do we obtain it? Hint: contemplate the words "superposition" and "homogeneous".

☞ The general solution $x(t)$ of a driven, damped oscillator is the sum of the particular solution $x_P(t)$ for driving force $f_0 \cos(\omega t)$ <u>plus</u> the homogeneous solution $x_H(t)$ for zero driving force. What behaviour does this produce? The homogeneous solution corresponds to an *un*-driven damped oscillator, so it will <u>die away</u> as time $t \to \infty$. (You can see that from the exponential-decay term $e^{-\beta t}$ in front of $x_H(t)$, which is reproduced in the footnote.) The terms of the homogeneous solution are therefore called **transients**: they are needed at finite time to match whatever initial conditions you were given, but after a sufficiently long time, they fade away to zero, leaving only the particular solution, $x_P(t)$, oscillating forever at the driving frequency ω.

(e) So, if we wait long enough, only the particular solution $x_P(t) = A\cos(\omega t - \delta)$ survives. The amplitude A you obtained in part (b) contains some interesting behaviour! Consider the case where the driving frequency ω is fixed, but you can adjust the natural, undamped frequency ω_0 of the oscillator. A classic example is a radio: radio waves drive the circuit, and if you want to tune in a particular station of frequency ω, you adjust the tuner circuit's natural frequency $\omega_0 = 1/\sqrt{LC}$ by turning a dial. What value of ω_0 should you choose to obtain the <u>maximum</u> amplitude A from the radio circuit? (Just stare at your formula for A and you will see it.) This is called the **resonant frequency** of the circuit, for a particular driving frequency.

(f) Adjusting your system so that it is at resonance produces, by definition, the response of the largest amplitude. Interestingly, the adjustment is a bit different if it is the driving frequency ω you can adjust, not ω_0. Most mechanical systems fall in this category: you can't adjust the parameters of a playground swing, for example, so ω_0 is fixed, but you *can* select the frequency ω at which you *push* the swing. Find the resonant driving frequency ω_{RES} that maximizes A if ω_0 is fixed. This time, the answer is not obvious, so you must take a derivative.

☞ Note that the resonant frequencies are <u>about the same</u> when the oscillator is very **weakly damped**: when $\beta << \omega_0$, we find $\omega_{0,RES} = \omega$ and $\omega_{RES} \approx \omega_0$. This is the most common situation as one typically tries to *minimize* damping in an oscillating system!

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[3] (a) A driven oscillator's steady-state solution will <u>not</u> have any damping and will oscillate at the <u>driving</u> frequency $\rightarrow \tilde{\Omega} = i\omega$.
(b) $A = f_0 / \sqrt{(\omega_0^2 - \omega^2) + (2\beta\omega)^2}$ and $\tan\delta = 2\beta\omega / \left(\omega_0^2 - \omega^2\right)$. (c) None! (d) What you just found is the **particular solution**, $x_P(t)$, that solves the inhomogeneous equation with driving force $f(t) = f_0 \cos(\omega t)$. It has no free parameters. To obtain the general solution, you add in the **homogeneous solution**, i.e. the general solution to the undriven EOM $\ddot{x} + 2\beta\dot{x} + \omega_0^2 x = 0$. That is the damped-oscillator solution we obtained in class: $x_H(t) = e^{-\beta t}\left(\tilde{A}_+ e^{t\sqrt{\beta^2 - \omega_0^2}} + \tilde{A}_- e^{-t\sqrt{\beta^2-\omega_0^2}}\right)$. It has the free parameters you need.
(e) $\omega_{0,RES} = \omega$ (f) $\omega_{RES} = \sqrt{\omega_0^2 - 2\beta^2}$