Physics 325 – Homework #8 **due in 325 homework box by Fri, 1 pm**

All solutions must clearly show the steps and/or reasoning you used to arrive at your result. You will lose points for poorly written solutions or incorrect reasoning. Answers given without explanation will not be graded: **"NO WORK = NO POINTS"**. *However you may always use any relation on the 3D-calculus and 1D-math formula sheets without proof. Please write your* **NAME** *and* **DISCUSSION SECTION** *on your solutions.*

Equilibrium : When can a purely *U*-based analysis be used?

In class, we discussed two methods for finding a system's **equilibrium points** and determining their **stability**:

- Method **A**: analyze the first and second derivatives of the **potential energy** $U(q)$
- Method **B**: use the system's **EOMs** to find the equilibrium points, then obtain their approximate solution **near** equilibrium (in the style of a **small-oscillation analysis**) to determine stability.

The "U-based" method A is faster, but it *doesn't always work*. Below is a summary of the conditions under which it does work. First, we need $T+U$ to be conserved. The reason is that $\dot{T}+\dot{U}=0$ is the only way we know *so far* to build an EOM using only energies. We don't need any EOM to *use* method A, but we do need an EOM to *prove* the conditions below ... so for now, Method A is confined to $T+U$-conserving systems.

A purely U-based equilibrium analysis is possible if all of these conditions are met:

(1) Potential energy must have this form: $U(q)$ + optional irrelevant constant
i.e. U depends only on q, not $\dot{q}$ or t. This form is required for $T+U$ conservation, so nothing new here.

(2) Kinetic energy must have this form: $T(q,\dot{q})=\frac{1}{2}\mu(q)\,\dot{q}^2$ + optional irrelevant constant.
The function $\mu(q)$ plays the role of an "effective mass" that may depend on the position / angle of your system. A perfect example is the falling & sliding cube from Homework 7: using the cube's rotation angle ϕ as our coordinate q, the cube's kinetic energy is $T(\phi,\dot{\phi})=\frac{1}{2}\dot{\phi}^2\left(I'+\frac{1}{2}Mb^2\sin^2\phi\right)$. The term in parenthesis is the effective mass function $\mu(\phi)$.

(3) The effective mass μ in the kinetic energy must ***behave like a normal mass*** in these ways:

- $\mu(q)$ and its derivative $\mu'(q)$ must be finite (no singularities!) at all q that might be equilibrium points.
- $\mu(q)$ must be positive (just like a normal mass) at all equilibrium points.

Of course, if μ is just a constant mass, as in $T=\frac{1}{2}m\dot{x}^2$, these conditions are trivially satisfied.

The proof is at the end of this homework. If all this seems too complicated, there's always Method B: brute force analysis of the system's motion. It never fails. ☺ But don't give up on pure U-based equilibrium analysis; it can speed things up tremendously if you know U or can calculate it easily, and the conditions are not that complicated once you get used to them. Also, the forms of U and T needed for this method are *very common*. We will revisit them in a couple of weeks when we have more powerful energy-based tools at our disposal.

Problem 1 : The Power of *U*

To become familiar with the conditions that allow a U-based equilibrium analysis, let's perform such an analysis using the EOM-based Method B, which always works. Consider a completely general system that has one independent coordinate q, potential energy $U(q)$, kinetic energy $T(q,\dot{q})=\frac{1}{2}\mu(q)\,\dot{q}^2$, and conserves $T+U$.

$U(q)$ and $\mu(q)$ are known but unspecified functions. We will perform an EOM-based equilibrium analysis of this generic system and thereby derive the elements of the U-based Method A. You may find it helpful to consult the proof in the Appendix: this problem follows that proof, it just starts with simpler forms for T & U.

(a) Task #1 with an EOM-based equilibrium analysis is of course to write down the EOM. Use energy conservation to write down the EOM in terms of U & μ, their derivatives U' & μ', the coordinate q, and / or the coordinate's derivatives $\dot{q}$ & $\ddot{q}$. You should find that all of your terms have a common factor that you can cancel. (This common factor *always* appears when you obtain an EOM from $\dot{T}+\dot{U}=0$.)

(b) Task #2 is to find the system's equilibrium point(s) $\bar{q}$. These are the points where $q(t)=\bar{q}$ (a constant) solves the EOM. Plug in this simple solution to find the **condition on U that identifies an equilibrium point**. That's the first part of Method A : finding the equilibrium points using only $U(q)$.

FYI: One of our Method A conditions is that $\mu(q)$ and $\mu'(q)$ must be finite wherever there might be an equilibrium point. Suppose there is an equilibrium point at some value $\bar{q}$, and suppose the mass function has a singularity there: $\mu(\bar{q})=\infty$. Do you see how such a singularity could ruin the equilibrium condition $U'(\bar{q})=0$? $\rightarrow$ The term $\mu(\bar{q})\,\ddot{q}$ = infinity times zero, which does *not* have to be zero!

(c) Task #3 – last one – is to determine the stability of any equilibrium point(s) $\bar{q}$ you found. In our EOM-based method, we do this with a small-oscillation analysis : we *solve* the EOM for points *very close to* $\bar{q}$ by using Taylor approximations wherever possible. Here's a recap of the steps from last week's discussion:

- Shift your coordinate : replace q with $\varepsilon \equiv q-\bar{q}$ so you can Taylor-expand around $\varepsilon=0$ instead of $q=\bar{q}$. We have no actual functions to work with, so this is trivial: rewrite your EOM replacing $U(q)$ with $U(\varepsilon)$ (same for the other functions of q), and replacing $q, \dot{q}, \ddot{q}$ with the new coordinate ε and its derivatives.
- Taylor-expand each function of ε around $\varepsilon = 0$, keeping only the constant and linear terms.
- Identify the lowest non-vanishing order of ε that appears in your EOM and drop all terms of higher order. Treat $\dot{\varepsilon}$ and $\ddot{\varepsilon}$ as being of the same order as ε. (See the next page for a discussion of this point.)

The result is the leading-order EOM for points very near $q=\bar{q}$ ($\varepsilon=0$). You will immediately recognize the ODE and its possible solutions, $\varepsilon(t)$. If the equilibrium point $\bar{q}$ near which you are working is stable, you will get an *oscillating* solution; if it is unstable, you will get a *runaway* solution. Using your leading-order EOM, determine the **conditions on U that respectively identify stable and unstable equilibrium** at an equilibrium point $\bar{q}$. That's the second part of Method A : analyzing stability using only $U(q)$.

FYI: Another of our Method A conditions is that the effective mass μ must be positive at any equilibrium point. Do you see what happens if $\mu(\bar{q})$ is *negative*? $\rightarrow$ The signs of your stability conditions are reversed!

(d) Assuming that $q=\bar{q}$ is a point of stable equilibrium, find a formula for the system's small-oscillation frequency ω around that point. Your formula should involve only $U(q)$, $\mu(q)$, and/or their derivatives, evaluated at $\bar{q}$. This is an extremely useful formula $\rightarrow$ if Method A works, you can obtain the small-oscillation frequency using only T and U. No Taylor approximations or EOM solutions required. ☺

(e) Return to your Taylor-expanded EOM from part (c) after truncation to lowest non-vanishing order. Suppose we had an equilibrium point where $U''\big|_{\varepsilon=0}=0$. What solution $\varepsilon(t)$ do you get?

FYI: This case doesn't fit into either of our two stability categories. It is called **neutral equilibrium**. The term is hard to find in any textbook; the only place I've found it is in Goldstein (the venerable graduate text) where, as usual, he offers colourful alternatives: "**labile, neutral,** or **indifferent equilibrium**". The Free Dictionary online actually has a superb definition: http://www.thefreedictionary.com/Neutral+equilibrium. To be exact, the situation $U''\big|_{q=\bar{q}}=0$ you just considered gives you **neutral equilibrium to lowest order**. Higher-order derivatives of U must be consulted to determine the system's ultimate stability, but $U''|_{q=\bar{q}} = 0$ means the system can remain *approximately* at rest *near* its equilibrium state for a non-negligible amount of time. True neutral equilibrium occurs when U has no dependence at all on a particular coordinate; then it can remain at any value of that coordinate forever.

A Frequently Asked Question : Are $\dot{\varepsilon}$ and $\ddot{\varepsilon}$ really of the same order as ε ?

For testing stability near equilibrium, we can make $\varepsilon \equiv q - \bar{q}$ as small as we like … and we can *also* make $\dot{\varepsilon}$ and $\ddot{\varepsilon}$ as small as we like. Let me prove it to you. A system's behaviour near equilibrium will be sinusoidal if it's stable or exponential if it's unstable. In the stable case, we have $\varepsilon(t) = \varepsilon_0 \sin(\omega t + \delta)$, where the **amplitude** ε_0 is very small (compared to $\bar{q}$, or to any other parameter in the problem with the same dimensions as ε). The derivatives $\dot{\varepsilon}(t) = \varepsilon_0 \omega \cos(\omega t + \delta)$ and $\ddot{\varepsilon}(t) = -\varepsilon_0 \omega^2 \sin(\omega t + \delta)$ are proportional to that same tiny amplitude ε_0, so they can be made as small as we want, just like ε itself. For unstable equilibrium, we have $\varepsilon(t) = \varepsilon_0 \left(e^{\omega t} + \delta e^{-\omega t} \right)$. As in the sinusoidal case, the derivatives simply pull out one or two factors of ω; $\dot{\varepsilon}$ and $\ddot{\varepsilon}$ are still proportional to that tiny amplitude ε_0. This allows us to treat ε **and** $\dot{\varepsilon}$ **and** $\ddot{\varepsilon}$ **as arbitrarily small** and **of the same order**. It is somewhat disturbing to be treating quantities with *different units* as being of the same order, but it is legitimate. The best way to think about the equivalence is to realize that the small quantity whose order we care about is really the **amplitude** ε_0 in front of ε and all its derivatives.

Problem 2 : Bead on a Parabolic Wire

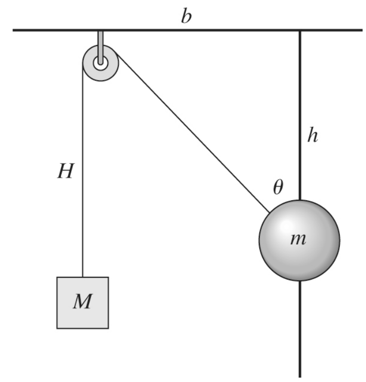
A bead of mass m is threaded onto a frictionless wire that is bent into the shape of a parabola: $z = x^2/2b$ and $y = 0$, where $+z$ points upward (opposite to uniform gravity). The bead performs oscillations along the wire, from $x = -a$ to $x = +a$.

(a) Show that the period, τ, of these oscillations is given by the integral $\tau = \frac{4}{\sqrt{gb}} \int_0^a \sqrt{\frac{b^2 + x^2}{a^2 - x^2}}\, dx$.

Guidance : **Conservation Laws & 1st-order ODEs**. You can proceed with this problem by finding the EOM via $\vec{F} = m\vec{a}$ or via $\dot{T} + \dot{U} = 0$, as usual. (About $\dot{T} + \dot{U} = 0$: if $T + U$ is conserved – as it clearly is in this problem – you can *always* recover one force-based EOM by taking the time-derivative of $T + U =$ constant E. This procedure is essentially the *inverse* of our proof that $\vec{F} = m\vec{a}$ for a conservative force leads to $\dot{T} + \dot{U} = 0$.) If you calculate $\dot{T} + \dot{U} = 0$ or $F_x = m\ddot{x}$, you will get an EOM that is second-order in time. However …
There is an alternative way to use energy conservation. If the problem supplies enough boundary / initial conditions to determine the total energy E at any point in time, E will have that same value at all other points in time (it's conserved!), so you can use $T + U = E$ directly as your energy-based EOM. This equation is first-order in time: it depends on coordinates x and velocities $\dot{x}$ but not on accelerations $\ddot{x}$. A first-order ODE offers significant advantages over a second-order ODE: it is usually easier to solve … and even if it's not (which sometimes happens), you can simply take its time-derivative to obtain the corresponding second-order ODE.

(b) Now consider the small-oscillation case $a \ll b$. Apply your approximation skills to the integrand to determine τ to lowest non-vanishing order in the small quantity $a/b \ll 1$. (Your answer should have two terms.) The integral table on our 1DMath formula sheet has everything you need.

Problem 3 : A Contraption

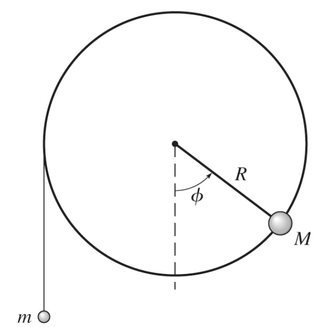


A ball of mass m with a hole through it is threaded on a frictionless vertical rod. A massless string of length l is attached to the ball and runs over a massless, frictionless pulley that is placed a horizontal distance b from the rod. The other end of the string supports a block of mass M, as shown in the figure. As the positions of both masses are determined by the angle θ, this problem may look multi-dimensional but it is not as it has only one independent coordinate.

(a) Find the potential energy $U(\theta)$ of the two masses under the influence of uniform gravity g in the downward direction.

(b) Find out whether the system has any <u>equilibrium positions,</u> $\bar{\theta}$, and determine what <u>constraint</u> must be imposed on the masses m and M for equilibrium to be possible at all. Finally, determine the <u>stability</u> (i.e. stable or unstable) of any equilibrium positions you found.

Problem 4 : Another Contraption



A massless wheel of radius R is mounted on a frictionless horizontal axle. A point mass M is glued to the edge of the wheel and a mass m hangs from a string wrapped around the perimeter of the wheel.

(a) Determine the total potential energy $U(\phi)$ of the two masses under the influence of uniform, downward-pointing gravity g. For definiteness, adjust your expression so that $U(\phi)=0$ when $\phi=0$. (Fixing the overall value of U to some particular value at some reference point makes no physical difference, of course, but the suggested choice will make the ensuing calculations more convenient.)

(b) Determine what <u>constraint</u> must be imposed on the masses m and M so that the system has at least one equilibrium position. Assuming that this constraint holds, find the <u>equilibrium positions</u>, $\bar{\phi}$, (major hint: there are <u>two</u> of them) and determine their <u>stability</u> (stable or unstable equilibrium).

(c) To generate a warm happy feeling, evaluate the torque on the wheel at the equilibrium angles $\bar{\phi}$ you just found and thereby double-check that the system is indeed at equilibrium.

(d) Use wolframalpha.com or your fancy graphing calculator or phone app to plot $U(\phi)$ for these two cases: $m = 0.7\ M$ and $m = 0.8\ M$ over the range $-\pi < \phi < \pi$. Sketch the shape of the curves on your answer sheet then use them to describe the behavior of the system in each case if it is <u>released from rest at $\phi = 0$</u>.
Note: You should check that your curves make sense by looking for the *two equilibrium points* that you found in part (b) ➔ they should appear clearly as two local extrema on both of your $U(\phi)$ curves. In addition, the *stability-of-equilibrium* that you found for these points in part (b) should also be apparent : any point of stable equilibrium will be a local *minimum* of $U(\phi)$ and any point of unstable equilibrium will be a local *maximum*.

(e) Write down an expression that can be solved for the <u>critical value</u> of the ratio m / M that determines whether the system oscillates or "runs away" (unwinds itself) if it is released from rest at $\phi = 0$. Your expression will not be solvable analytically, but you can use a computer or fancy calculator to solve it numerically; to check your work, you should find a critical value of $m / M = 0.725$.

HINTS: This question has everything to do with part (d); if you haven't completed it, this question makes no sense. Once you *do* have your plots and have interpreted them, you will see that the condition for the critical value of m / M is the situation *in between* your $m / M = 0.7$ and $m / M = 0.8$ plots where the system will <u>*just barely*</u> maintain an oscillating behavior if released from rest at $\phi=0$. Final hint: You must translate those words into a *condition on $U(\phi)$ at one of your two equilibrium angles* $\bar{\phi}$. (Which one? … you'll figure it out. ☺)

Problem 5 : Hemisphere Balanced on a Hemisphere

A solid hemisphere of radius b has its flat surface glued to a horizontal table. A second solid hemisphere of different radius a rests on top of the first one so that the curved surfaces are in contact. The surfaces of the hemispheres are rough (so no slipping occurs between them), and both hemispheres have uniform mass distributions. The two objects are in equilibrium when the top one is "upside down", i.e. with its flat surface parallel to the table but above the hemispheres' contact point. Show that this equilibrium position is stable if $a < 3b/5$.

Suggestion: For this is problem, it is *really* worth checking the conditions on the introductory page to see if the U-based "method A" might work. If it does, your stability analysis will be greatly simplified.

Proof : Conditions for U-based Equilibrium Analysis (One Coordinate, $T+U$ conserved)

As described in the introduction, we restrict our attention to systems where $T+U$ is conserved. We will also only consider systems with 1 independent coordinate, which we label q. (We will be able to move past these restrictions once we have the new tools we'll be learning in the coming weeks.) Our goal is to discover what restrictions we must place on the forms of T and U to allow a **purely U-based analysis** of a system's **equilibrium points**, where U' gives you the equilibrium points and U'' gives you their stability. We're calling that "Method A"; it is extremely efficient, but it doesn't always work. In problem 1, you start with the specific forms $U(q)$ and $T(\dot{q},q) = \frac{1}{2}\mu(q)\dot{q}^2$, then prove that Method A works for such a system. This appendix shows that those specific forms are *required* for Method A to work, i.e. that there are no other possibilities. This proof follows the same steps as Problem 1, we just start with the most general possible forms of T and U, and figure out as go along what terms and dependences we must remove.

Here are the most general allowed forms for the potential and kinetic energies of a $T+U$-conserving system:

$$U(q) \quad \text{and} \quad T(\dot{q},q,t) = \tfrac{1}{2}\mu(q,t)\,\dot{q}^2 + \alpha(q,t)\,\dot{q} + \beta(q,t)$$

U can only dependent on q because of energy conservation (see proof from lecture). T will be at most quadratic in the "velocity" $\dot{q}$ because it ultimately comes from the formulae $T = \frac{1}{2}m(\dot{x}^2+\dot{y}^2+\dot{z}^2)$, $T = \frac{1}{2}I\dot{\phi}^2$, $T = \frac{1}{2}m(\dot{s}^2+s^2\dot{\phi}^2+\dot{z}^2)$, etc ..., all of which are quadratic in $\dot{q}_i$ because they are originated from the Cartesian expression. The term $\beta(q,t)$ with *no* dependence on $\dot{q}$ is included to take into account possible constraints on other coordinates of the system. For example, the rotating spring from lecture included a time-dependence constraint, $\phi = \Omega t$, that produces $T = \frac{1}{2}m(\dot{s}^2+s^2\Omega^2)$. The $s^2\,\Omega^2$ piece would appear in our $\beta(s,t)$ term as it has no dependence on $\dot{s}$ at all. Concerning the α term: it is essentially impossible to generate a linear dependence on velocity, but it's easier to include the $\alpha(q,t)\,\dot{q}$ term and prove that we must remove it for Method A than to prove it is impossible to generate.

Part 1 : Finding equilibrium points with U'

The first element of Method A is that **equilibrium points occur at extrema of $U(q)$**. More rigorously:

$$U'\big|_{q=\bar{q}} = 0 \text{ is a necessary and sufficient condition for } \bar{q} \text{ to be an equilibrium point.}$$

We must discover what restrictions (if any) we must impose on U and/or T to make this true. Our strategy is the same as in Problem 1: analyze the equilibrium using the infallible Method B. In this method, we build the system's equation of motion (EOM) then search for **solutions of the form $q(t)$ = a constant**, which we call $\bar{q}$. Any values $\bar{q}$ that work are equilibrium points: if we place the system at rest at $q = \bar{q}$, and $q(t) = \bar{q}$ is a solution of the EOM, then the system will <u>remain</u> at $q = \bar{q}$ … which is the definition of equilibrium. To build our EOM from U and T, we use energy conservation and the multi-dimensional chain rule:

$$0 = \frac{d}{dt}\left[U(q) + T(\dot{q},q,t)\right] = \frac{dU}{dq}\dot{q} + \frac{\partial T}{\partial \dot{q}}\ddot{q} + \frac{\partial T}{\partial q}\dot{q} + \frac{\partial T}{\partial t} \quad \text{and} \quad T(q,\dot{q},t) = \tfrac{1}{2}\mu(q,t)\,\dot{q}^2 + \alpha(q,t)\,\dot{q} + \beta(q,t)$$

$$\text{so} \quad \left[-\frac{dU}{dq}\right]\dot{q} = \left[\mu\dot{q} + \alpha\right]\ddot{q} + \left[\frac{\partial \mu}{\partial q}\frac{\dot{q}^2}{2} + \frac{\partial \alpha}{\partial q}\dot{q} + \frac{\partial \beta}{\partial q}\right]\dot{q} + \left[\frac{\partial \mu}{\partial t}\frac{\dot{q}^2}{2} + \frac{\partial \alpha}{\partial t}\dot{q} + \frac{\partial \beta}{\partial t}\right]$$

The condition we seek is on $U' = dU/dq$, so factor out the $\dot{q}$ that appears in all but two of the terms:

$$-\frac{dU}{dq} = \left[\mu\ddot{q} + \frac{\partial \mu}{\partial q}\frac{\dot{q}^2}{2} + \frac{\partial \alpha}{\partial q}\dot{q} + \frac{\partial \beta}{\partial q} + \frac{\partial \mu}{\partial t}\frac{\dot{q}}{2} + \frac{\partial \alpha}{\partial t}\right] + \frac{1}{\dot{q}}\left[\alpha\ddot{q} + \frac{\partial \beta}{\partial t}\right] \tag{1}$$

This is our equation of motion. We now plug in the equilibrium solution $q(t) = \bar{q}$, which has $\dot{q} = 0$ and $\ddot{q} = 0$:

$$\begin{aligned}-\left.\frac{dU}{dq}\right|_{q=\bar{q}} &= \left[\mu\cancel{\ddot{q}} + \frac{\partial \mu}{\partial q}\frac{\cancel{\dot{q}}^2}{2} + \frac{\partial \alpha}{\partial q}\cancel{\dot{q}} + \frac{\partial \beta}{\partial q} + \frac{\partial \mu}{\partial t}\frac{\cancel{\dot{q}}}{2} + \frac{\partial \alpha}{\partial t}\right]_{q=\bar{q}} + \frac{1}{\cancel{\dot{q}}}\left[\alpha\cancel{\ddot{q}} + \frac{\partial \beta}{\partial t}\right]_{q=\bar{q}} \\ &= \left[\frac{\partial \beta}{\partial q} + \frac{\partial \alpha}{\partial t}\right]_{q=\bar{q}} + \frac{1}{0}\left[(\alpha)(0) + \frac{\partial \beta}{\partial t}\right]_{q=\bar{q}}\end{aligned} \tag{2}$$

For $U'(\bar{q}) = 0$, i.e. the existence of a potential energy extremum at $\bar{q}$, to be a necessary and sufficient condition for $\bar{q}$ to be an equilibrium point, the equilibrium solution $q(t) = \bar{q}$ we just plugged in must satisfy the EOM above if and only if the left-hand side is zero. Thus, all four terms remaining on the right-hand side must be zero at any equilibrium point. Since we do not *know* the equilibrium points in advance, the $U'(\bar{q}) = 0$ method for equilibrium hunting is only of practical use if those terms are zero at *all* values of q. First, α must be completely removed, as it is multiplied by an undefined 0/0 that could take on any value. Second, the partial derivatives $\partial\beta/\partial q$ and $\partial\beta/\partial t$ must be zero; since β can only depend on q and t, it can only be a constant. With α removed and β reduced to a constant (call it β_0), the allowed form of kinetic energy is greatly simplified:

$$T(q,\dot{q},t) = \tfrac{1}{2}\mu(q,t)\,\dot{q}^2 + \beta_0 \tag{3}$$

We already argued that the α term – the one linear in $\dot{q}$ – is impossible to generate anyway, so its removal is no loss. However we have also lost the α term, and that does restrict the class of problems to which we can apply Method A. Further, we must inspect equation (2) more carefully. Those slashes indicate the quantities $\dot{q}$ and $\ddot{q}$ that are zero in an equilibrium solution ... but the terms they belong to may *not* be zero if their coefficients are *infinite!* We must therefore impose these restrictions on the surviving term $\tfrac{1}{2}\mu(q,t)\dot{q}^2$:

- μ and its derivatives $\dfrac{\partial \mu}{\partial q}$ and $\dfrac{\partial \mu}{\partial t}$ must be **finite** at any point that might be an equilibrium point

Part 2 : **Analyzing stability with U''**

The second element of Method A is the use of the **curvature** of $U(q)$ to determine the stability of an equilibrium point. The rule is: **stable (unstable) equilibrium occurs when $U''(q)$ is positive (negative)**. To test this rule, we return to Method B and perform a small-oscillation style analysis of the system's motion near one of its equilibrium points, $\bar{q}$.

First, we <u>shift coordinates</u> to the equilibrium point by introducing the variable $\varepsilon \equiv q - \overline{q}$. Our equilibrium position is at the "origin", $\varepsilon = 0$, of this new coordinate, which greatly simplifies our work. Changing variables from q to ε is trivial:

$$\varepsilon \equiv q - \overline{q} \quad \rightarrow \quad q = \varepsilon + \overline{q} \quad \rightarrow \quad dq = d\varepsilon \,, \quad \dot{q} = \dot{\varepsilon} \,, \quad \ddot{q} = \ddot{\varepsilon}$$

Using the restricted form of kinetic energy from equation (3), our EOM from equation (1) also becomes much smaller. Here is the new EOM in the shifted coordinate system:

$$-\frac{dU}{dq} = \mu \ddot{q} + \frac{\partial u}{\partial q}\frac{\dot{q}^2}{2} + \frac{\partial \mu}{\partial t}\frac{\dot{q}}{2} \quad \rightarrow \quad -\frac{dU(\varepsilon)}{d\varepsilon} = \mu(\varepsilon,t)\ddot{\varepsilon} + \frac{\partial u(\varepsilon,t)}{\partial \varepsilon}\frac{\dot{\varepsilon}^2}{2} + \frac{\partial \mu(\varepsilon,t)}{\partial t}\frac{\dot{\varepsilon}}{2} \qquad (4)$$

We want to *solve* this equation for values of ε very close to 0, so we Taylor-expand the functions in our EOM. (That's why I made the dependences of the functions explicit in equation (4).) At the end, we only keep leading non-vanishing order in $\varepsilon, \dot{\varepsilon}, \ddot{\varepsilon}$ so we will only expand each function to first order: $f(\varepsilon) \approx f(0) + \varepsilon\, f'(0)$.

$$-\left.\frac{dU}{d\varepsilon}\right|_{\varepsilon=0} - \varepsilon \left.\frac{d^2U}{d\varepsilon^2}\right|_{\varepsilon=0} \approx \ddot{\varepsilon}\left(\mu(0,t) + \varepsilon \left.\frac{\partial \mu}{\partial \varepsilon}\right|_{\varepsilon=0}\right) + \frac{\dot{\varepsilon}^2}{2}\left(\left.\frac{\partial \mu}{\partial \varepsilon}\right|_{\varepsilon=0} + \varepsilon \left.\frac{\partial^2 \mu}{\partial \varepsilon^2}\right|_{\varepsilon=0}\right) + \left.\frac{\dot{\varepsilon}}{2}\frac{\partial \mu}{\partial t}\right|_{\varepsilon=0}$$

First, the far-left term, $-U'(0)$, is zero $\rightarrow$ the entire goal of part 1 was to preserve the rule that U' is necessarily zero at any equilibrium point, like $\varepsilon = 0$. With that removed, we hunt through the rest of the terms for the <u>lowest non-vanishing order</u> in $\varepsilon, \dot{\varepsilon}, \ddot{\varepsilon}$. That order is 1 and it appears in three places: term #2 = $-\varepsilon\, U''(0)$, term #3 = $\ddot{\varepsilon}\,\mu(0,t)$, and the far-right term = $\frac{1}{2}\dot{\varepsilon}\,\partial\mu/\partial t|_{\varepsilon=0}$. All the remaining terms are of second or third order and can be tossed. The resulting EOM is

$$-\varepsilon\, U''(0) \approx \ddot{\varepsilon}\, \mu(0,t) + \left.\frac{\dot{\varepsilon}}{2}\frac{\partial \mu}{\partial t}\right|_{\varepsilon=0} \qquad (5)$$

Let's neglect the far-right term for the moment by assuming that μ has no explicit time-dependence. The first two terms are the EOM for a simple harmonic oscillator:

$$\ddot{\varepsilon} \approx -\varepsilon \frac{U''(0)}{\mu(0)}$$

This most familiar of equations will have an oscillating solution if $U''/\mu > 0$ at the equilibrium point, and a "runaway" exponential solution if $U''/\mu < 0$. That precisely matches the curvature-of-U rule we are trying to reproduce … *as long as μ is positive*. Further, μ must be *strictly* positive, i.e. not equal to zero. (A zero value would produce a singularity!) Thus we have our next condition for Method A to work:

- μ must be **positive** at any equilibrium point, just like a normal mass

Finally, what do we do about the third term in equation (5)? It represents an explicit time-dependence in the effective mass function $\mu(q, t)$. From your differential equations course, you probably recognize this $\dot{\varepsilon}$ term as the *damping* term that is added when an oscillator is subject to velocity-dependence drag forces like air resistance. Damping will not turn a stable oscillation into an unstable exponential ... unless the sign of $\partial\mu/\partial t$ is *negative*. (That would produce "reverse" damping = Boom!) We could add yet another restriction ... but if the behavior of the system is this complicated, one really should use Method B and closely inspect what is going on. So, we will simply restrict our effective mass terms to have no explicit time-dependence and stop trying to extend the validity of Method A any further.