Physics 325 – Homework #9 **due in 325 homework box by Fri, 1 pm**

All solutions must clearly show the steps and/or reasoning you used to arrive at your result. You will lose points for poorly written solutions or incorrect reasoning. Answers given without explanation will not be graded: **"NO WORK = NO POINTS"**. *However you may always use any relation on the 3D-calculus and 1D-math formula sheets without proof; both are posted in the same place you found this homework. Finally please write your* **name** *and* **DISCUSSION SECTION** *on your solutions.*

Summary of Variational Calculus: If you want to extremize a quantity S that is integrated over some path $\{q_i(t)\}$ of your system between fixed endpoints, and S is described by the integral

$$\boxed{S = \int L(q_i(t), \dot{q}_i(t), t)\, dt} \quad \text{with fixed endpoints } t_1, q_i(t_1), \text{ and } t_2, q_i(t_2),$$

then the path $\{q_i(t)\}$ that extremizes this integral satisfies the **Euler-Lagrange equation**

$$\boxed{\frac{\partial L}{\partial q_i} = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right)} \quad \text{for each coordinate } q_i$$

That's it! Working with these equations boils down to identifying which parameters of your system are:

- the **independent variable** t that serves as your parameter of integration,
- the **(remaining) coordinates** $q_i(t)$ describing the state of the system (I say "remaining coordinates" since the variable of integration, t, is often chosen to be one of the coordinates)
- the integrand $L(q_i, \dot{q}_i, t)$, called the **Lagrangian** of the problem

Solving calculus-of-variations problems is thus a 3-step game:

1. Set up the integral S you are trying to extremize (if isn't just given to you).
2. Play "match the letters" → what's "t"? what are the "q_i? what's "L"?
3. Write down the E-L equation for each q_i, then solve them for the path $\{q_i(t)\}$ that extremizes S.

> **You may use http://wolframalpha.com (or any equivalent tool) to evaluate any of the integrals on this homework.** You must set up the integrals yourself, but some are complicated enough that it's more helpful for you to practice using integration software than to get hints about what substitutions to make (as no such hints are generally provided in Real Life ☺).
> **Please read the Appendix : Mathematica Oddities**, at least before you do Problems 4 or 5.

Problem 1 : Simple Euler-Lagrange

Find the path $y(x)$ for which the integral $\int \sqrt{x}\sqrt{1+\dot{y}^2}\, dx$ is stationary (i.e. is extremized) between fixed endpoints, and sketch its shape.

DOT-NOTATION NOTE: Since the coordinates of your path (q_i) depend on only one parameter (t) it is common to use dot notation to denote the derivative $\dot{q}_i \equiv dq_i/dt$ even if your independent variable is not time. If you feel uncomfortable with this, feel free to rewrite the integral in this problem as $\int \sqrt{x}\sqrt{1+y'^2}\, dx$.
The point is : there is no possible confusion between y' and $\dot{y}$ since y is only a function of one variable = the one independent variable you're using as your integration parameter. (A path *always* has only one parameter of integration, and the Euler-Lagrange equations only apply to path integrals.)

Problem 2 : Fermat's Oasis

The air above a hot road is less dense near the road and more dense as you get higher above the road. This results in a refractive index that varies with the height y above the road : $n(y) = n_0\sqrt{1+y/a}$ where n_0 and a are constants and y=0 is the road. Fermat's principle states that the path light travels between two fixed points is the path that minimizes the total travel time. Obtain the Euler-Lagrange equation for the paths that light can take in this heated air. Instead of finding its general solution, show that the particular path $y(x) = x^2/4a$ solves it.

FYI: This path is a parabola that touches the road. It shows how light from the blue sky can "bounce" off the hot road and hit your eyeball while you are walking along. Your eye will interpret this as something blue on the road off in the distance, which is the familiar "oasis mirage" of water apparently lying ahead on a hot surface.

STRATEGY: This problem illustrates an important strategic issue that arises all the time in variational problems: <u>what independent variable should you choose</u>? In this problem you are asked to find $y(x)$, so the choice seems obvious: the independent variable "t" is x and the dependent coordinate "q" is y. But there is an alternative! You may instead pick y as your "t", x as your "q", use the E-L equation to solve for $x(y)$ ... and then invert your result to obtain the requested path $y(x)$. The choice you make is surprisingly important! Here's why:

E-L equation: $$\frac{\partial L}{\partial q} = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right)$$ ← right-hand side is scary!

The right-hand side is usually a *horrible mess* to compute, and it produces an ODE that is *second-order* in t (i.e., contains $\ddot{q}$). The equation is greatly simplified if you can arrange things so that <u>***q*** **does not appear in** ***L***</u>. That makes the left-hand side *zero*, and the equation becomes $\partial L/\partial \dot{q} = C$ (a constant). This is a *first-order* ODE, which is usually much simpler to solve. As we discussed in lecture, any coordinate q_i that does not explicitly appear in the Lagrangian is called a **cyclic** or **ignorable coordinate**. Of course there are always exceptions. If you want $y(x)$, but decide to solve for $x(y)$ instead to simplify the E-L equation, the final step of *inverting* $x(y)$ may occasionally be so tricky that the original procedure may actually be easier.

The Fermat's Oasis problem is a perfect opportunity for you to learn this technique. I urge you to try both alternatives: use x as your independent variable, see what Lagrangian you get, and set up the E-L equation … then repeat the procedure using y as your independent variable. In one case you will get a Lagrangian of the form $L(q,\dot{q})$ and in the other case you will get the form $L(\dot{q},t)$. See which E-L equation is easier to solve! ☺

This strategy issue appears again in the later problems, especially in problem 4 where good strategy is *essential*.

Problem 3 : Maximum Enclosed Area

You are given a string of fixed length l with one end fastened at the origin, and you are to place the string in the xy-plane with its other end on the x axis in such a way as to enclose the <u>maximum area</u> between the string and the x-axis. Show that the required shape is a semicircle.

GUIDANCE #1 — E-L SETUP: The area enclosed is of course $A = \int y\,dx$ but this form of integral *cannot* be subjected to the Euler-Lagrange equations because the *endpoints in x are not fixed!* Remember: the E-L equations only apply when the integral we are trying to extremize has <u>fixed endpoints</u> in both the <u>independent variable</u> "t" and the <u>dependent coordinates</u> "q_i". The quantities that *are* fixed are the length l of the string and the starting and ending y-coordinates. We must thus change variables from (x, y) to (s, y) where s is <u>cumulative distance along the path</u>. This s runs from 0 to l, and y runs from 0 back to 0, so we now have fixed endpoints in both coordinates, good! Depending on which coordinate you choose as your independent variable, your integral to maximize will thus have the form $\int_0^l L(y,\dot{y},s)\,ds$ or $\int_0^0 L(s,\dot{s},y)\,dy$. (If you are bothered by the 0-to-0 limits on that second form, see the footnote[1].) All you must do is translate dx from the original area integral into a form involving ds and dy. Relating dx, dy, and ds to each other is easy once you realize that "ds" is the little step you take along your path when you change position by dx and dy … which is exactly the line element of x,y-space. To be precise, it is the magnitude of the line element: $ds = dl \equiv \left| d\vec{l} \right|$. Once you have recast your area integral in terms of y and s, you can write down the E-L equation for the maximal-area path $y(s)$ (or $s(y)$).

GUIDANCE #2 — E-L SOLUTION: After you have obtained the E-L equation for $y(s)$, you have to solve it. The simplest route is probably to use the information we were given: that the solution is a semicircle. Guessing the solution form in advance is, after-all, one of the primary techniques for solving ODEs: "Guess and Plug"! ☺ To prove that a semicircle is the solution – which is your task – you must first figure out what $y(s)$ is for a semicircle of fixed circumference l. (Hint: introduce a temporary angle, express s and y in terms of it, then get rid of it.) Once you've figured out $y(s)$ for the semicircle, check explicitly that it satisfies your Euler-Lagrange equations. If it does, your proof is complete. ☺ But read a little further ...

Warning: you must NEVER use prior knowledge or intuition about what the final path will look like when SETTING UP the integral to extremize! When you set up $A = \int y\ dx = \int L(y, \dot{y}, s)\ ds$, you must not use any relations that restrict $y(s)$ to a semi-circle, or any other shape → $y(s)$ must be *completely free to vary* in any way it wants to between the fixed endpoints. If not, you have *completely destroyed* the entire machinery that leads to the Euler-Lagrange equations. If this point is not 100% clear, please ask!

Problem 4 : Geodesic on a Cone

(a) Consider a conical surface whose apex is at the origin, whose axis of symmetry runs along the $+z$ axis, and whose half-angle is α. Using cylindrical coordinates, calculate the path $s(\phi)$ of minimum distance between the endpoints $(s, \phi) = (s_0, -\pi/2)$ and $(s_0, +\pi/2)$ where s_0 is a positive constant.

(b - **not for points**: **optional part** that is pretty cool ☺) Imagine that you constructed this conical surface from a rolled-up piece of paper. Let's return the paper to its original flat shape: mentally slice the paper cone with scissors along the line $\phi = \pi$, then unwrap the paper and flatten it on your desk. The resulting shape will look like Pac-Man. ☺ Figure out the transformation that maps the cone coordinates s and ϕ onto the 2D-polar coordinates r and θ on your piece of paper. (Hint: the angle ϕ has to be *rescaled*.) Finally, express your geodesic in terms of the paper coordinates r and θ and show that it is actually a *straight line* on the paper! ☺

Problem 5 : Cost-Effective Flying

An airplane flies in the (x,z)-plane, with z=0 being ground level and $+z$ pointing upward. The plane flies from $(x,z) = (-a, 0)$ to $(+a, 0)$. The density of the air decreases with altitude, so fuel usage is reduced at higher z … but of course, going to higher z also increases the total length of the plane's path. What a perfect situation for mathematical optimization! Given that the cost of flying the plane is e^{-kz} per unit distance traveled along the plane's path, find the flight path $z(x)$ that minimizes the total cost of the flight. (Assume that the constant k is positive and that $ka < \pi/2$.) Important: Please read the Appendix. Depending on what route you take, you may get an integral of a similar form to those discussed in point (2).

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[1] If the choice $S = \int_0^0 L(s, \dot{s}, y)\ dy$ bothers you because the endpoints in y are the same, realize that you **never** actually **do** this integral. This is important: the **values of the endpoints** make **no appearance whatsoever** in the setup of the Euler-Lagrange equations. All that matters is that the endpoints are *fixed*, otherwise you are solving a completely different sort of problem that is outside the applicability of the E-L equations. The E-L equations are the ODEs that specify the *complete set of paths* that extremize S between *any* fixed endpoints. Once you have obtained the E-L ODEs and found their general solution, *then* you can apply any specific endpoints you have, as boundary conditions, to obtain the specific path that passes through those endpoints. If you now go back and *perform* that integral S over a path that runs from y=0 to 0, you would certainly manipulate it (e.g. split it into two parts) to avoid the trivial 0 answer … but that is just a technical issue concerning a particular integral over a particular path.

APPENDIX – MATHEMATICA ODDITIES

As you probably know, wolframalpha.com is a user-friendly front-end to Mathematica, the industry-standard software package for analytical math. The best way to learn how to use WolframAlpha is to head straight for the Examples pages. There's no need to read any manual: unlike Mathematica, the WolframAlpha front-end is incredibly flexible in the syntax it accepts and almost anything you type (or mistype!) will be correctly interpreted. Now about the back-end. The Mathematica software behind WolframAlpha is extremely powerful, but it has some idiosyncrasies that you should be aware of. Here are two that appear often, and that you are particularly likely to encounter in Problem 5.

(1) Mathematica doesn't simplify expressions as much as you would.

Here's an example of WolframAlpha / Mathematica behaving strangely:

Input: `integrate sqrt(a/(a-x^2)) dx`

Output: $$\int\sqrt{\frac{a}{a-x^2}}\,dx \;=\; \sqrt{\frac{a}{a-x^2}}\sqrt{a-x^2}\,\tan^{-1}\!\left(\frac{x}{\sqrt{a-x^2}}\right) + \text{constant}$$

!?! The program *refuses* to cancel those $\sqrt{a-x^2}$ terms in the numerator and denominator! This oddity arises because Mathematica keeps *very* careful track of all possible values that the arguments of special functions can have (e.g., those square roots might have negative arguments, making the results imaginary). It is also considering the possibility of *complex values* for all variables. To illustrate, here's what you have to do in full-blown Mathematica to get it to perform an "obvious" cancellation:

```
In[24]:= Integrate[Sqrt[1 / (1 - x^2)], x]
```

Out[24]= $\sqrt{\frac{1}{1-x^2}}\;\sqrt{1-x^2}\;\text{ArcSin}[x]$

```
In[21]:= Integrate[Sqrt[1 / (1 - x^2)], x, Assumptions → {x ∈ Reals , Abs[x] < 1}]
Out[21]= ArcSin[x]
```

You have to add *both* of those "Assumption" clauses to get the cancellation. I haven't found a way to impose Assumption clauses in WolframAlpha, so the message is: keep your eyes open for terms you can cancel.

(2) Mathematica's output can change drastically with slight input changes … and watch out for *i*'s !

Here is what WolframAlpha does with an integral handed to it in two totally equivalent forms:

Form 1: $$\frac{1}{2}\int\sqrt{\frac{a\,e^x}{1-a\,e^x}}\,dx \;=\; \frac{e^{-x/2}}{\sqrt{a}}\sqrt{\frac{a\,e^x}{1-a\,e^x}}\sqrt{1-a\,e^x}\;\sin^{-1}\!\left(\sqrt{a}\,e^{x/2}\right) \;+\; \text{constant}$$

Form 2: $$\frac{1}{2}\int\sqrt{\frac{a}{e^{-x}-a}}\,dx \;=\; \frac{e^{-x/2}}{\sqrt{a}}\sqrt{\frac{a}{e^{-x}-a}}\sqrt{a\,e^x-1}\;\ln\!\left(a\,e^{x/2}+\sqrt{a}\sqrt{a\,e^x-1}\right) \;+\; \text{constant}$$

This happens a lot: a seemingly insignificant change in the input can produce substantial changes in the output. (For an example related to point 1, try `integrate 1/sqrt(1-x^2)` and `integrate sqrt(1/(1-x^2))` … !?!) The reason is the same as before: Mathematica is accounting for values of the variables that we don't care about (e.g. complex values), so slight changes in input form can remove or add possibilities that we don't need to consider. In this case, however, we really seem to have a problem: does this integral give an arcsin or a log?? To figure it out, you must first address Issue #1 → there are many "obvious" cancellations you can perform:

Reduced Form 1: $\sin^{-1}\left(\sqrt{a}\,e^{x/2}\right)$ + const Reduced Form 2: $\sqrt{\dfrac{a-e^{-x}}{e^{-x}-a}}\,\ln\left(a\,e^{x/2}+\sqrt{a}\sqrt{a\,e^{x}-1}\right)$ + const

Now look at that Form 2: the first factor is $\sqrt{-1}=i$! To be exact, it's ±*i*, as every square root has an implicit ± accompanying it. Are these expressions really equivalent?? Let's investigate further by defining $u \equiv \sqrt{a}\,e^{x/2}$:

Form 1: $\sin^{-1}(u)$ + const Form 2: $\pm i\ln\left(u+\sqrt{u^2-1}\right)\pm i\ln(\sqrt{a})$ + const

The second term in Form 2, $\pm i\ln(\sqrt{a})$, can be absorbed into the constant of integration … and if you are very experienced with complex numbers, you can now see that these forms *are* the same. Here are some relations that are frequently useful for massaging Mathematica's unpredictable output into a more helpful form:

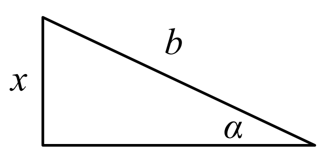
☞ $\arcsin(u)=-i\ln\left(u+\sqrt{u^2-1}\right)+\dfrac{\pi}{2}$ $\arccos(u)=i\ln\left(u+\sqrt{u^2-1}\right)$ $\arctan(u)=\dfrac{i}{2}\ln\left(\dfrac{1-iu}{1+iu}\right)$

They are all easy to prove with Euler's theorem (see Unit 13 from PHYS 225 for tons more about that), and you can find them all in Wikipedia (just search for arcsin and you'll get the whole article), but they are not very pleasant to work with. So what's the final message? Something like this:

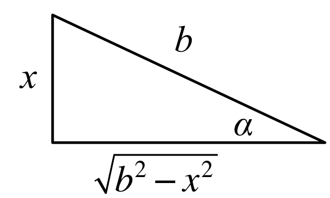
☞ If Mathematica gives you an expression that has an **imaginary *i*** in it and your problem doesn't involve complex numbers at all, there is undoubtedly an **equivalent real-valued expression**. To find it, first try **reformatting your input**, e.g. by moving terms from numerators to denominators, or breaking up / combining square roots … and if that fails, try the relations above. (In my experience it's usually one of those.)

(3) (Arc-)Trig Triangles

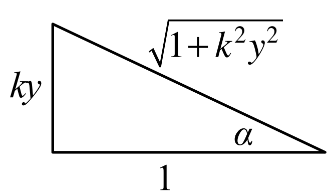
Suppose you perform an integral using Mathematica (or an integral table, or by hand) and the result is $\tan^{-1}(x/a)$ … but you have some reason to expect or want a *different* arc-trig function, e.g. $\cos^{-1}$. No problem: <u>you can always translate any trig or arc-trig function into any other</u>. Suppose you know that $\sin(\alpha)=x/b$, but to complete your calculations, you need $\tan(\alpha)$. You could bash away with trig relations and algebra, but there's a better way → **the most powerful way to represent any trig result is with a triangle**.
Sine is opposite-over-hypotenuse, so writing $\sin(\alpha)=x/b$ is completely equivalent to drawing this triangle:



You can now invoke Pythagoras to fill in the length of the remaining side →



That triangle allows you to read off all other trig functions: $\cos(\alpha)=\sqrt{b^2-x^2}\,/\,b$, $\tan(\alpha)=x/\sqrt{b^2-x^2}$, etc.



You can do the exact same thing with arc-trig functions. Suppose Mathematica gives you $\alpha=\tan^{-1}(ky)$. That statement is completely summarized by the triangle at left … which again allows you to read off whatever you want:

$\alpha=\tan^{-1}(ky)=\sin^{-1}\left(ky/\sqrt{1+k^2y^2}\right)=\cos^{-1}\left(1/\sqrt{1+k^2y^2}\right)$, etc.