Physics 325 – Homework #10 **due in 325 homework box by Fri, 1 pm**

Important: midterm 2 is coming & we want to post solutions, so **NO LATE ASSIGNMENTS** will be accepted.

Lagrangian Mechanics : A Practical Summary

The reason we studied the calculus of variations is that it has a profound application in physics: the recasting of mechanics in terms of a variational principle. This approach is known as **Lagrangian Mechanics** or, more generally, **Analytical Mechanics**[1]. Here is a 5-step summary of the Lagrangian approach. We will prove all this in the coming week.

Step 1: Figure out the number of **degrees of freedom (DOF),** ***n*** of the system.

Step 2: Select the ***n*** **generalized coordinates** $\{q_1, q_2, \ldots q_n\}$ you will use to describe your system.

Step 3: Calculate the **Lagrangian** $\boxed{L(q_i, \dot{q}_i, t) = T - U}$ of the system.

Step 4: Apply the **Euler-Lagrange equations** to your Lagrangian to get the system's ***n*** **equations of motion**

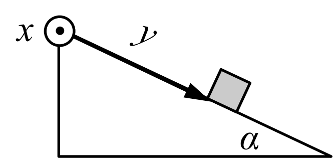
$$\boxed{\frac{\partial L}{\partial q_i} = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right)} \quad \text{for each generalized coordinate } q_i$$

When the E-L equations are applied to the Lagrangian of a mechanical system (rather than to some random mathematical quantity, as in last week's calculus-of-variations problems), they are referred to as simply the **Lagrange equations** for the system. Poor Professor Euler is thus banished to the realm of pure mathematics. Also remember that you can always drop additive and multiplicative constants from any Lagrangian without affecting the Lagrange equations; a glance at the equations' form makes this obvious.

Step 5: **Solve the EOMs** (equations of motion) to determine the **system's behavior** $q_1(t), q_2(t), \ldots, q_n(t)$.

Now we must PRACTICE this powerful, force-free approach to solving mechanics problems! The problems below and in this week's discussion contain strategy notes to help you master this new style of mechanics. The strategy notes are placed immediately after the question part they refer to.

Problem 1 : Mass on a Tilted Plane



Consider a mass m moving on a frictionless plane that slopes at an angle α with respect to the horizontal. Uniform gravity g points in the downward direction.

(a) Write down the Lagrangian in terms of these two generalized coordinates: x measured horizontally across the slope, and y measured down the slope.

CONSTRAINT EQUATIONS: These x and y coordinates are perpendicular, so they form a nice Cartesian coordinate system where z points in the direction normal to the plane. The **constraint equation** for this system — the equation that reduces the number of independent coordinates from 3 (x,y,z) to 2 (x,y) — is simply $z=0$ (the particle stays on the plane). Do you have to write it down? When it's this simple, no: the constraint just removes z from the problem entirely, which is *exactly* the type of simplification you're looking for when you choose your generalized coordinates. When you write down the kinetic energy, you do have to at least *remember* the $z=0$ constraint to get $T = \frac{1}{2}m\left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2\right)$ in terms of $\dot{x}$ and $\dot{y}$ only.

(b) Find the two Lagrange equations of motion and show that they are exactly what you would expect from a force-based $\vec{F} = m\vec{a}$ analysis.

---

[1] The term **Analytical Mechanics** is a bit more general in that it includes another formulation of this approach called "Hamiltonian Mechanics". We'll come to that, but its basis is identical to that of Lagrangian mechanics, and it is no better for doing calculations.

Problem 2 : Particle on a Cone

A particle of mass m is confined to move on the surface of a cone with its axis on the z axis, with its vertex at the origin and pointing "down" (i.e. the cone is on the $+z$ side), and with half-angle α (i.e. this is the angle the sides make with the $+z$-axis). The particle's position can be specified by two generalized coordinates, which you can choose to be the coordinates s and ϕ of the cylindrical coordinate system.

(a) Write down the equations that give the three Cartesian coordinates (x, y, z) of the particle in terms of the generalized coordinates (s, ϕ), and vice versa.

CONSTRAINT EQUATIONS AGAIN: This time, you *must* write down the constraint equation describing the geometric condition that the particle is confined to the surface of the cone as it is *not* as trivial as z=0.

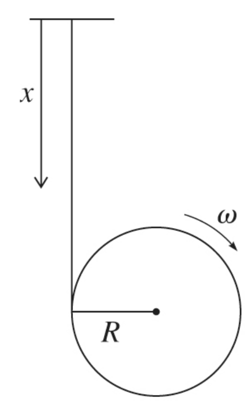
(b) Write down the Lagrangian for the particle assuming that it is under the influence of no external forces like gravity, just the constraint forces keeping it on the cone.

(c) Write down the Lagrange equation for the generalized coordinate ϕ and show that it immediately tells us that *some component* (which one?) of *some type of momentum* (which type?) is conserved.

(d) Now add uniform gravity g in the $-\hat{z}$ direction to your Lagrangian. As you can quickly see, this change does not affect the equation of motion you found in (c). With your new Lagrangian, write down the second Lagrange equation of motion : the one for the generalized coordinate s. It cannot be solved analytically, but show that it allows the solution $s = s_0$ (constant) as long as the angular velocity $\dot{\phi}$ has a specific value. Find that value in terms of the given parameters s_0, m, g, and/or α.

DECOUPLING EOMs: In problems with more than one degree of freedom ($n > 1$), the n equations of motion you get will generally be **coupled differential equations**: more than one of the functions $q_i(t)$ you're trying to find will show up in each equation. You can hardly ever make any progress toward a solution until your equations of motion are **decoupled**, i.e. turned into **separated** EOMs that each involve only one coordinate q_i (and/or its derivatives). If you have any **conserved quantities** , a.k.a. **constants of motion**, they are invaluable tools for decoupling your EOMs. A rich source of constants of motion are cyclic coordinates q_i: if you have any, evaluate their EOMs *first* and leave them in the 1st-order form " $\partial L / \partial \dot{q}_i$ = constant C". These expressions can often be used to replace the cyclic coordinates q_i and/or their derivatives $\dot{q}_i$ in the *other* EOMs with constant expressions, and this if often enough to decouple the remaining EOM(s). Be careful not to make the mistake of decoupling your equations in one part, then going *back* to the coupled version in a later part! (It's easy to do when things get complicated.)

Problem 3 : A YoYo



The figure shows a simple model of a yoyo. A massless string is suspended vertically from a fixed point and the other end is wrapped several times around a uniform solid cylinder of mass M and radius R. When the cylinder is released it moves vertically down under the influence of uniform gravity g, rotating as the string unwinds.

(a) Write down the Lagrangian, using the distance x shown in the figure as your generalized coordinate. You can look up any moments of inertia you need without proof (also for the rest of these problems.)

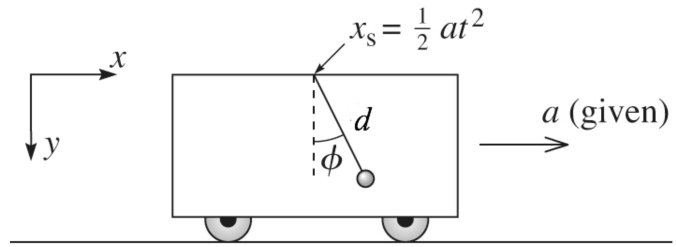
(b) Find the Lagrange equation of motion, show that the cylinder accelerates downward with constant acceleration $\ddot{x}$, and determine the value of $\ddot{x}$.

Problem 4 : Bead on a Spiral Wire

A smooth wire is bent into the shape of a helix (a spiral) as follows : the wire's perpendicular-distance from the z-axis is R (constant) and the spiral "climbs" a distance b in the $+z$ direction on each turn. A bead of mass m is threaded onto the wire, and uniform gravity g points in the $-z$ direction. Using z as your generalized coordinate, write down the Lagrangian for the bead, obtain the EOM, and determine the bead's vertical acceleration $\ddot{z}$.

Problem 5 : Pendulum in a Car

The pendulum shown in the figure is suspended from a fixed point inside a railroad car that is being forced to accelerate with constant acceleration a. The pendulum consists of a mass m placed at the end of a rigid massless rod of length d and is only allowed to swing in the xy plane. Uniform gravity g points downwards ($+y$ direction in the coordinate system shown).



FORCED-MOTION CONSTRAINTS: The phrase "**<u>forced to <do something></u>**" is code for "some mysterious force whose nature you *don't have to worry about* maintains the <indicated motion>". The constant acceleration a of the car is simply one of the constraints on the problem. The force that's causing it should be treated exactly like a constraint force: it is whatever it needs to be to maintain this acceleration, and it does not appear in your Lagrangian as a contribution to U but simply as a condition on the variables. The figure illustrates this by handing you the (obvious) position of the pendulum's suspension point: $x_s(t) = \frac{1}{2}at^2$.

(a) Write down the Lagrangian for the system and determine the EOM for the angle ϕ. To tidy up your EOM, rewrite the linear combination $A\sin\theta + B\cos\theta$ in the alternate form $C\sin(\phi + \alpha)$.

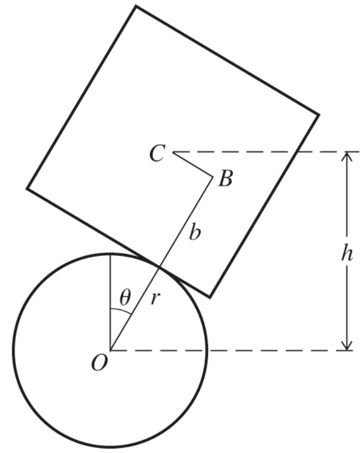
FINDING COMPLICATED VELOCITIES: When several moving things are attached to each other, finding the velocities you need for T can be tricky. Here is the safest method to find v^2 for a particular object:
(1) Write down the <u>position coordinates</u> of the object using one of our <u>standard coordinate systems</u>. (You will need a *velocity formula*, and we only *have* such formulae in Cartesian, cylindrical, and spherical coordinates.)
(2) <u>Differentiate those position coordinates with respect to time</u> and combine the results to obtain v^2.
(3) Finally, switch back to your generalized coordinates.
Using a **temporary coordinate system** like this is very common in Lagrangian mechanics. Just remember step (3): once you're done, everything must be expressed entirely in terms of your generalized coordinates, their time-derivatives, and/or time, or the Lagrangian framework will be ruined. (We will prove this in lecture.)

(b) Find the equilibrium angle ϕ_0 at which the pendulum can remain fixed (relative to the car).

(c) What is the frequency of small oscillations about the equilibrium angle ϕ_0?

(d) Calculate the Hamiltonian H for this system, then answer these two questions: (i) Is the Hamiltonian <u>conserved</u>? (ii) Is the Hamiltonian equal to <u>$T+U$</u> ? Briefly explain why or why not for each of these questions using the principles we discussed in lecture 10B.

Problem 6 : Cube Balanced on a Fixed Cylinder



A wooden cube of side $2b$ and mass M is balanced on a fixed rubber cylinder of radius r, with uniform gravity g pointing downwards. The cube can't slip on the cylinder's rubber surface but it can roll from side to side. Assuming that $b < r$, use the Lagrangian approach to find the frequency of small oscillations about the equilibrium position $\theta = 0$. You may obtain any moments of inertia you need from Wikipedia or WolframAlpha without proof. Also, one strategic suggestion: make your small-angle approximations to the Lagrangian *before* you construct the Lagrange equation.

Problem 7 : Pendulum–Spring

A pendulum is made from a massless spring (force constant k and unstretched length l_0) that is suspended at one end from a fixed pivot O and has a mass m attached to its other end. The spring can stretch and compress but cannot bend, and the whole system is confined to a single vertical plane.

(a) Write down the Lagrangian for the pendulum, using as generalized coordinates the usual angle ϕ for a pendulum (the angle with respect to the direction of gravity = downward) and the length s of the spring, then find the system's two Lagrange equations.

(b) These EOMs cannot be decoupled or analytically solved in general. However they *can* be solved for small oscillations around the equilibrium position $\{ \phi = 0, s = l \}$. Here, l is <u>not</u> the unstretched length l_0 of the spring; it is a constant whose value you must figure out. By analyzing the small-oscillation motion of the system, figure out the small-oscillation frequencies for both the ϕ and s coordinates.