Physics 325 – Homework #12 **due in 325 homework box by Fri, 1 pm**

This homework consists of typical Ph.D. qualifying exam problems, just with a bit more guidance than on the "qual" exam. Use all the techniques you have learned, especially those concerning ...

- **Conserved quantities** (a.k.a. **constants of motion**): *always* keep an eye out for them.
  - They are enormously valuable in turning coupled differential equations into separated ones.
  - They give you 1st-order EOMs that *may* be easier to solve than the 2nd-order Lagrange equations.
  - If you are given boundary conditions (e.g. the usual "position & speed at $t = 0$"), you can use them to determine the value of a conserved quantity instead of just leaving it as an undetermined constant. This is a great way to inject boundary-condition information into your solution.
- You can often figure out conserved quantities by **physical intuition** rather than by cyclic coordinates. The prime example is **energy conservation**: if you realize in advance that $T+U$ is conserved, just apply it. There is no need to build the Hamiltonian (which can be time-consuming) unless time is cyclic but energy is not conserved, and you really need another conserved quantity (e.g. if your Lagrange equations seem unsolvable). Also remember that if you have one DOF, you only need one EOM; if energy is conserved, "$T+U$ = constant" has the same advantages as the Lagrange equations (energy-based, no constraint forces to worry about, etc).
- Energy conservation also has a unique advantage over the Lagrange EOMs, namely the second appearance of **effective potential** we discussed in Lecture 12A. The slick U_{EFF} method of equilibrium analysis cannot be applied to the Lagrangian if you have more than one DOF ... but it *can* be applied to "$T+U$ = constant" under certain conditions.

To demonstrate how helpful energy conservation can be, even in Lagrangian mechanics, and to remind you about effective potential, here is a homework-wide hint:

At least one part of each question can be simplified by using energy conservation and/or U_{EFF} ...
and in a couple of cases, the use of energy conservation is **essential**.

Problem 1 : Spherical Pendulum

The "spherical pendulum" is just a simple pendulum that is free to move in any sideways direction. (By contrast, the unqualified word "pendulum" or the explicit phrase "plane pendulum" implies that motion is confined to some vertical plane.) The bob of a spherical pendulum moves on a spherical surface centered on the pendulum's support point; the length of the pendulum is the radius of this surface. A convenient choice of coordinates is spherical (r, θ, ϕ) with the origin at the point of support and the z axis pointing *down* = in the direction of uniform gravity. The angles θ and ϕ are a good choice of generalized coordinates.

(a) Find the Lagrangian and the two Lagrange equations of motion.

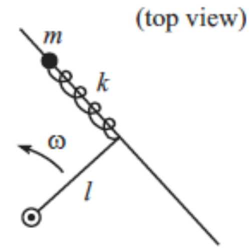
(b) Explain what the ϕ equation tells us about the z component of angular momentum, l_z. As always, it's not enough to guess, you must explicitly show how the quantity you get from your ϕ equation is related to l_z.

(c) For the special case that ϕ = constant, describe what the θ equation tells us. To be precise, what simple system does the spherical pendulum reduce to in this situation?

(d) Back to the general case! Use the ϕ equation of motion to replace $\dot{\phi}$ by l_z (and other terms, of course) in the θ equation, thereby constructing a separated equation for $\theta(t)$ alone. Use this equation to determine an expression for the angle θ_0 at which θ can remain constant. Why is this motion called a "conical pendulum"?

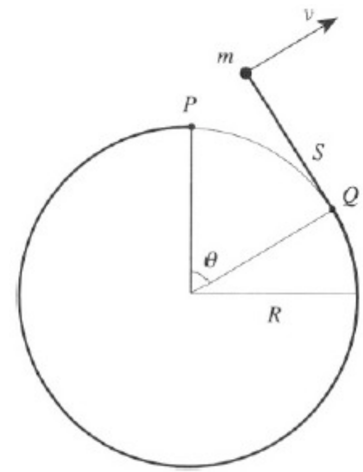
(e) Using whatever method you like, show that if $\theta(t) = \theta_0 + \varepsilon(t)$ with ε very small, then θ oscillates around θ_0 in harmonic motion. Find the frequency of these small oscillations in terms of θ_0, g, and R.

Problem 2 : Spring on a T



A rigid "T" consists of a long rod glued perpendicular to another rod of length l that is pivoted at the origin. The "T" rotates around in a horizontal plane with constant frequency ω. A mass m is free to slide along the long rod and is connected to the intersection of the rods by a spring with spring constant k and relaxed length zero. Calculate $r(t)$, where r is the position of the mass along the long rod. You should get three solution forms, depending on the relative values of ω^2 and k/m.

Problem 3 : A String Unwinds

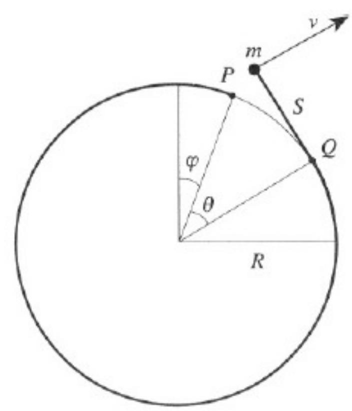


A point mass m is attached to a thin, massless string whose other end is attached to a fixed cylinder of radius R at the point P. Initially ($t \leq 0$), the string is tightly wound all the way around the cylinder, with the point mass m located at point P on the cylinder. Then, at $t = 0$, an impulse is delivered to the point mass so that it acquires a velocity v_0 in the outward radial direction. (i.e. The impulse kicks that mass straight upward from point P). This starts the unwinding of the string. The figure shows the state of the system at a random moment $t > 0$, with s denoting the length of string that has been unwound and Q indicating the point where the string loses contact with the cylinder. No external forces act on the system.

(a) Find the Lagrangian for the mass m in terms of the generalized coordinate θ, then solve for $\theta(t)$ using whatever method you like. Remember to apply the given initial conditions to your solution: at $t = 0$, θ is 0 and the velocity of the mass is v_0. If you encounter a differential equation you can't solve, feel free to use wolframalpha.com.

(b) Using your $\theta(t)$ solution, find the angular momentum of the mass m around the cylinder's axis of symmetry (which is perpendicular to the page). Is angular momentum conserved? Why or why not?

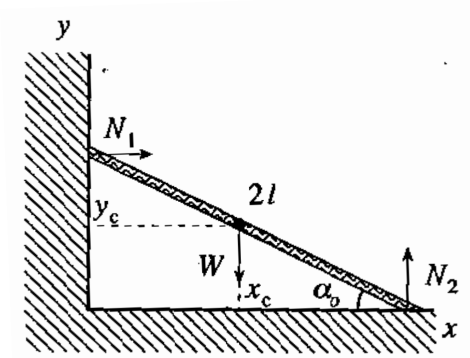
(c) Write down the total energy of the system and determine if it is conserved.



(d) **NEW SITUATION**: The cylinder is now **free to rotate** without friction around its axis of symmetry. Take the cylinder have mass M and to be hollow with walls of negligible thickness. (It thus has the word's simplest moment of inertia. ☺) Introduce the angle ϕ shown in the second figure to denote the cylinder's rotation angle relative to its initial ($t = 0$) orientation. Calculate the Lagrangian for the [cylinder + point mass] system in terms of the generalized coordinates θ and ϕ, and use the symbol $\alpha \equiv (1+M/m)$ to simplify your expressions.

(e) Identify two conserved quantities and write them in terms of θ and ϕ. Finally, solve for $\theta(t)$. (Hint: those conserved quantities will help you!) Which way does the cylinder rotate: clockwise or counter-clockwise?

Problem 4 : A Sliding Ladder



A ladder of mass m and length $2l$ stands against a frictionless wall with its feet on a frictionless floor. It is released from rest at time t=0 with initial angle α_0. Calculate the angle α^* at which the ladder loses contact with the wall (i.e. when the normal force N_1 goes to zero). You may assume that the ladder loses contact with the wall before it loses contact with the floor. Hint: read the note on page 1 about injecting boundary condition information.