

Rotating Frame Pseudo-Forces: $\vec{f}_{cf}^* = (\vec{\Omega} \times \vec{r}^*) \times \vec{\Omega}$, $\vec{f}_{Cor}^* = 2\vec{v}^* \times \vec{\Omega}$, $\vec{f}_{azim}^* = \vec{r}^* \times \dot{\vec{\Omega}}$ where $\vec{f} \equiv \vec{F} / m$
 Cylindrical Coordinates: $\vec{v} = \dot{s}\hat{s} + s\dot{\phi}\hat{\phi} + \dot{z}\hat{z}$, $\vec{a} = \hat{s}[\ddot{s} - s\dot{\phi}^2] + \hat{\phi}[s\ddot{\phi} + 2\dot{s}\dot{\phi}] + \hat{z}[\ddot{z}]$

Problem 1 : Corrections to g for Stationary Objects

Let’s approximate the Earth as a perfect sphere of radius $R_e = 6,400$ km; in this approximation the g -value (acceleration due to gravity) is constant over the planet’s surface: $g \equiv f_{grav} \equiv F_{grav} / m = GM_e / R_e^2 = 10$ m/s². Of course the Earth also rotates, at an angular speed that we will label ω and treat as constant (since it *is* very nearly constant).

All of our experiences and measurements occur in the rotating reference frame of the Earth, so gravity is *not* the only force that acts on masses in our Earth-bound world: pseudo-forces must also be taken into account. The result is a modification of the acceleration-per-unit-mass we experience. It is typically called “**g-effective**” (or “effective- g ”) and gets the symbol g_{eff} ; effective- g reflects the fact that massive objects at the Earth’s surface experience an acceleration that is *not* $\vec{g} \equiv \vec{f}_{grav}$ but a slightly modified quantity $\vec{g}_{eff} \equiv (\vec{f}_{grav} + \vec{f}^*)$.

In this problem, we will consider only objects that are at rest in the Earth’s rotating reference frame. Thus, the only pseudo-force that will play a role in this problem is the centrifugal force.

(a) What is the magnitude of g_{eff} for an object at rest on the Earth’s surface? Express your answer as a function of the object’s polar angle θ relative to the North Pole and in terms of the known Earth constants g , ω , and R_e . The centrifugal force is a small effect at the Earth’s surface, so give your answer as an additive (or subtractive) correction to g that is accurate only to leading non-vanishing order in whatever small quantity you can find. For the small quantity: you need to figure out what it is, and you need to *show* that it’s small compared to 1. Finally, watch out: the forces involved may point in *different directions*.

(b) What is the largest magnitude of the fractional deviation $|g_{eff} / g - 1|$ on Earth, and where does it occur? (At the poles? the equator? in between?) It is good to know this number so that you can evaluate how important the centrifugal correction is in real life!

Problem 2 : Corrections to g for Moving Objects

As described in the previous problem, in a rotating frame like the Earth, the force of gravity is not the only force in play. You already analyzed the effect of the centrifugal pseudo-force on stationary objects. When a moving object is involved, however, the Coriolis pseudo-force must also be considered.

A mass is placed at the equator, raised to an altitude d above the surface of the Earth, then dropped from rest. All this is done by a person on Earth, so “rest” means “at rest in the Earth’s frame”. Let \vec{g}_d be the effective- g at the release point from which the mass is dropped. (Since the mass is at rest at the release point, \vec{g}_d is the quantity you calculated in problem 1(a), namely $\vec{g} + \vec{f}_{cf}^*$.) Now consider the moment when the object reaches the Earth’s surface (altitude 0) and let \vec{g}_0 be the effective- g at this landing point. At the landing point, the object has fallen a vertical distance d and has therefore picked up some speed, which brings \vec{f}_{Cor}^* into play. Your task is to calculate the total radial change in effective- \vec{g} — i.e. the quantity $(\vec{g}_0 - \vec{g}_d) \cdot \hat{r}$ — that is produced by the combined effect of the centrifugal and Coriolis forces, again approximated to leading non-vanishing order in whatever small quantities you can find. This is a fairly heavy problem, so let’s proceed in steps.

(a) Identifying the small quantities that you can use for Taylor approximations is crucial for this problem, let’s do that first. We know the size of all the parameters in our problem except for the altitude d from which our mass is dropped. We are trying to determine the approximate size of the centrifugal and Coriolis effects in

“everyday” situations, so let’s use a large but non-insane value of $d \rightarrow$ let’s assume that d is of order 10 km, which is roughly the altitude of long-distance commercial flights. First, use your work from problem 1 to show that $f_{cf} / f_{grav} \ll 1$ for an object dropped from a plane. Second, show that $f_{Cor} / f_{grav} \ll 1$ is also an excellent approximation. (For this part, you might like to remember *terminal velocity* and the *peregrine falcon*. ☺)

(b) Next, calculate the contribution to $(\vec{g}_0 - \vec{g}_d) \cdot \hat{r}$ from the centrifugal force alone, approximating to leading non-vanishing order in small quantities. Hint: your answer should only depend on ω and d .

(c) Finally, calculate the contribution to $(\vec{g}_0 - \vec{g}_d) \cdot \hat{r}$ from the Coriolis effect, then add the two together to get the combined result for $(\vec{g}_0 - \vec{g}_d) \cdot \hat{r}$. Hint: again, your answer should only depend on ω and d .

Guidance: This problem is very similar to the one we worked through in lecture 13B (also Taylor §9.8). As you will soon discover when you start the problem, the Coriolis effect on the radial component of g_{eff} is a **second-order effect**. Split the Coriolis force up into two pieces:

- Let $\vec{f}_{Cor}^{(1)}$ be the dominant piece; it is *not* in the radial direction. (Think about what happens as the mass begins to fall ... what is the direction of the resulting Coriolis force?)
- Let $\vec{f}_{Cor}^{(2)}$ be the radial component of the Coriolis force; as you will see when you think about it, it only exists because of the action of $\vec{f}_{Cor}^{(1)}$, so we call it a second-order Coriolis effect.

All the pseudo-forces are small, only gravity is not; the key to this problem is remembering that we only want the leading non-vanishing order Coriolis correction to g at the landing point. (There is no Coriolis effect at the release point of course.) Anything of order small-squared should be dropped immediately. We can thus proceed by **successive approximation**: calculate $\vec{f}_{Cor}^{(1)}$ first, using a speed calculated *only* from the one *non-small* force in the problem, then calculate $\vec{f}_{Cor}^{(2)}$ at the landing point, again approximating like crazy.

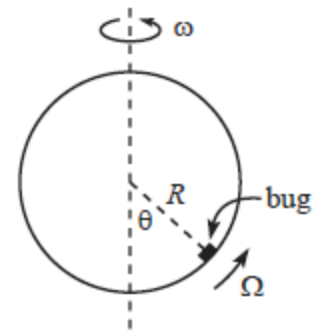
Problem 3 : Bug Walking on a Rotating Hoop

A circular hoop of radius R is made to rotate at constant angular speed ω around a diameter. A small bug walks at constant angular speed Ω around the hoop. Consider the force-per-unit-mass f that the hoop applies to the bug when the bug is at the angle θ shown in the figure, and let f_{\perp} be the component of f that is perpendicular to the plane of the hoop. Find the magnitude of f_{\perp} in two ways:

(a) Work in the inertial lab frame, i.e. treat this as a standard force or torque problem.

(b) Work in the rotating frame of the hoop, using your knowledge of pseudo-forces.

Advice about coordinate systems: It seems obvious that spherical coordinates are the way to go with this problem since the radial coordinate of the bug’s path is constant ($r = R$). However, in rotation problems, cylindrical coordinates are almost always the better choice, as the rotation itself is the key feature of the problem (i.e., the rotation is much tougher to deal with than the shape of the bug’s path). Align the z axis with the rotation axis, and you will find cylindrical coordinates ideally suited to almost any rotation problem. The velocity and acceleration formulae in cylindrical coordinates are provided at the top of page 1 for your convenience.



Problem 4 : Coriolis on a Space Habitat

qual exam problem

On Earth, a baseball player can hit a ball $d = 120$ m by giving it an initial angle of 45° to the horizontal. Take the acceleration due to gravity as $g = 10$ m/s². Suppose the batter repeats this exercise in a space “habitat” that has the form of a cylinder of radius $R = 10$ km and has an angular velocity about the axis of the cylinder sufficient to give an apparent gravity of g at radius R . The batter stands on the inner surface of the habitat (at radius R) and hits the ball in the same way as on Earth (i.e. at 45° to the surface and with the same speed), in a plane perpendicular to the axis of the cylinder. What is the furthest distance the batter can hit the ball, as measured along the surface of the habitat?

Guidance 1: The first part of this problem is clearly to figure out the initial conditions of the baseball on Earth, since you will use them on the Habitat for the second part of your calculation. This problem expects you to know that you should *ignore the centrifugal and Coriolis forces for the Earth trajectory* (not the trajectory on the station!) I think the authors expected you to know this from the fact that they gave all the numbers to only *two significant digits*, and you are supposed to know that the pseudo-forces caused by the Earth’s rotation are less than 1% of g . (Hence the importance of working through problems 1 and 2. ☺) Also, this implicitly means that your final answer concerning the trajectory on the station need only be accurate to two significant digits.

Guidance 2: In non-inertial frame problems, you always have a decision to make on the best strategy. Sometimes it is easier to do all calculations in an inertial frame where there are no pseudo-forces, then translate your findings to the accelerating frame; in other cases, it is easier to work directly in the non-inertial frame with all the pseudo-forces at work. (Problem 3 illustrated some of the differences you may encounter with these two approaches.) To gain experience, I encourage you to try both approaches here, but you will probably find the *first* approach (inertial-frame analysis) to be the easier one in this case.