

Useful formulae

Momentum of a particle:

$$\vec{p} = m \frac{d\vec{r}}{dt}$$

Force on a particle:

$$\vec{F} = \frac{d\vec{p}}{dt}$$

Center of mass:

$$\vec{r}_{cm} = \frac{1}{M} \int \vec{r} dm = \frac{1}{M} \sum_i m_i \vec{r}_i \quad (1)$$

Change in momentum of center of mass in presence of external forces F_i :

$$\frac{d\vec{p}_{cm}}{dt} = \sum_i \vec{F}_i \quad (2)$$

Cylindrical coordinates:

$$x = R \cos \phi \quad (3)$$

$$y = R \sin \phi \quad (4)$$

$$z = z \quad (5)$$

$$R = \sqrt{x^2 + y^2} \quad (6)$$

$$\vec{r} = x\hat{i} + y\hat{j} + z\hat{k} \quad (7)$$

$$\hat{R} = \cos \phi \hat{i} + \sin \phi \hat{j} \quad (8)$$

$$\hat{\phi} = -\sin \phi \hat{i} + \cos \phi \hat{j} \quad (9)$$

$$\frac{d\hat{R}}{dt} = \dot{\phi} \hat{\phi} \quad (10)$$

$$\frac{d\hat{\phi}}{dt} = -\dot{\phi} \hat{R} \quad (11)$$

$$\vec{v} = \dot{R} \hat{R} + R \dot{\phi} \hat{\phi} + \dot{z} \hat{z} \quad (12)$$

$$\vec{a} = (\ddot{R} - R\dot{\phi}^2) \hat{R} + (R\ddot{\phi} + 2\dot{R}\dot{\phi}) \hat{\phi} + \ddot{z} \hat{z} \quad (13)$$

$$\nabla^2 f = \frac{1}{R} \frac{\partial}{\partial R} \left(R \frac{\partial f}{\partial R} \right) + \frac{1}{R^2} \frac{\partial^2 f}{\partial \phi^2} + \frac{\partial^2 f}{\partial z^2} \quad (14)$$

Spherical coordinates (3D):

$$x = r \sin \theta \cos \phi \quad (15)$$

$$y = r \sin \theta \sin \phi \quad (16)$$

$$z = r \cos \theta \quad (17)$$

$$\vec{r} = x\hat{i} + y\hat{j} + z\hat{k} \quad (18)$$

$$\hat{\theta} = \cos \theta \cos \phi \hat{i} + \cos \theta \sin \phi \hat{j} - \sin \theta \hat{k} \quad (19)$$

$$\hat{\phi} = -\sin \phi \hat{i} + \cos \phi \hat{j} \quad (20)$$

$$\frac{d\hat{r}}{dt} = \dot{\theta} \hat{\theta} + \sin \theta \dot{\phi} \hat{\phi} \quad (21)$$

$$\frac{d\hat{\theta}}{dt} = -\dot{\theta} \hat{r} + \cos \theta \dot{\phi} \hat{\phi} \quad (22)$$

$$\frac{d\hat{\phi}}{dt} = -\dot{\phi} (\sin \theta \hat{r} + \cos \theta \hat{\theta}) \quad (23)$$

$$\vec{v} = \dot{r} \hat{r} + r \dot{\theta} \hat{\theta} + r \dot{\phi} \sin \theta \hat{\phi} \quad (24)$$

$$\vec{a} = (\ddot{r} - r\dot{\theta}^2 - r\dot{\phi}^2 \sin^2 \theta) \hat{r} + (r\ddot{\theta} + 2\dot{r}\dot{\theta} - r\dot{\phi}^2 \sin \theta \cos \theta) \hat{\theta} + (r\ddot{\phi} \sin \theta + 2\dot{r}\dot{\phi} \sin \theta + 2r\dot{\theta}\dot{\phi} \cos \theta) \hat{\phi} \quad (25)$$

$$\nabla^2 f = \frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial f}{\partial r} \right) + \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta} \left(\sin \theta \frac{\partial f}{\partial \theta} \right) + \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2 f}{\partial \phi^2} \quad (26)$$

Force from a potential:

$$\vec{F} = -\vec{\nabla} U \quad (27)$$

$$U(x) = - \int_{x_0}^x dx' F(x') \quad (28)$$

$$\frac{1}{2} m (v_2^2 - v_1^2) = \int_{x_1}^{x_2} dx' F(x') \quad (29)$$

Gradient of a scalar field:

$$\vec{\nabla} U = \frac{\partial U}{\partial x} \hat{i} + \frac{\partial U}{\partial y} \hat{j} + \frac{\partial U}{\partial z} \hat{k} \quad (30)$$

Curl of a vector field:

$$\vec{\nabla} \times \vec{F} = \hat{i} \left(\frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z} \right) + \hat{j} \left(\frac{\partial F_x}{\partial z} - \frac{\partial F_z}{\partial x} \right) + \hat{k} \left(\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y} \right) \quad (31)$$

Stokes' Theorem, which turns a line integral around a closed path into an integral over the enclosed area, with $d\vec{A}$ a vector pointing in the direction orthogonal to the area (for example, a

path that lies in the x - y plane gets turned into an area integral with $d\vec{A}$ pointing in the z direction, which picks out the z component of the curl):

$$\oint \vec{F} \cdot d\vec{r} = \int (\vec{\nabla} \times \vec{F}) \cdot d\vec{A} \quad (32)$$

Simple harmonic motion:

$$U(x) = \frac{1}{2}k(x - x_o)^2 + \text{constant} \quad (33)$$

$$F(x) = -k(x - x_o) \quad (34)$$

$$\omega \equiv \sqrt{\frac{k}{m}} \quad (35)$$

$$\text{If } \ddot{x} = -\omega^2 x \text{ then } x = A \sin(\omega t + \phi) \quad (36)$$

Taylor expansion:

$$\begin{aligned} f(x) \sim & f(x_o) + \frac{df}{dx}(x_o)(x - x_o) \\ & + \frac{1}{2!} \frac{d^2f}{dx^2}(x_o)(x - x_o)^2 \\ & + \frac{1}{3!} \frac{d^3f}{dx^3}(x_o)(x - x_o)^3 + \dots \end{aligned} \quad (37)$$

Rocket-related useful equation (for exhaust relative velocity \vec{u} ; careful with signs!):

$$\frac{d\vec{P}}{dt} = M \frac{d\vec{v}}{dt} + \vec{u} \frac{dm}{dt} \quad (38)$$

Center of mass frame scattering (relative velocity \vec{V}):

$$\vec{V} = \vec{v}_1 - \vec{v}_2 \quad (39)$$

$$\vec{p}_1 = \mu \vec{V} = -\vec{p}_2 \quad (40)$$

$$\mu = \frac{m_1 m_2}{m_1 + m_2} \quad (41)$$

Potential energy:

$$\text{Constant force : } U(x) = -Fx + C \quad (42)$$

$$\text{Spring : } U(x) = \frac{1}{2}kx^2 + C \quad (43)$$

$$1/r^2 \text{ Gravity : } U(x) = \frac{-GMm}{r} + C \quad (44)$$

Some useful approximations (for small arguments to first order):

$$\sin \theta \sim \theta - \dots \quad (45)$$

$$\cos \theta \sim 1 - \dots \quad (46)$$

$$\ln(1+x) \sim x - \dots \quad (47)$$

$$e^x \sim 1 + x + \dots \quad (48)$$

$$(1+x)^\alpha \sim 1 + \alpha x + \dots \quad (49)$$

Central force motion effective potential for reduced mass μ and angular momentum ℓ and potential $U(r)$:

$$U_{eff}(r) = U(r) + \frac{\ell^2}{2\mu r^2} \quad (50)$$

Orbit in a potential $U = -C/r$:

$$r(\theta) = \frac{(\ell^2/\mu C)}{1 - \sqrt{1 + (2E\ell^2/\mu C^2)} \sin(\theta - \theta_o)} \quad (51)$$

Taking $\theta_o = -\pi/2$, and using the following definitions:

$$r = \frac{r_o}{1 - \epsilon \cos \theta} \quad (52)$$

$$\epsilon \equiv \sqrt{1 + \frac{2E\ell^2}{\mu C^2}} \quad (53)$$

$$r_o \equiv \frac{\ell^2}{\mu C} \quad (54)$$

For a bound orbit, this is an ellipse, with the length of the longest axis A (all the way across i.e., the thing that would be the diameter, not the radius, for a circle) and the short axis B :

$$A = \frac{2r_o}{1 - \epsilon^2} = \frac{C}{-E} \quad (55)$$

$$B = \frac{2\ell}{\sqrt{-2\mu E}} \quad (56)$$

The longest and shortest distances from the central object:

$$r_{max} = \frac{r_o}{1 - \epsilon} \quad (57)$$

$$r_{min} = \frac{r_o}{1 + \epsilon} \quad (58)$$

Kepler's third law relating period T to length of long axis A :

$$T^2 = \frac{\pi^2}{2(M+m)G} A^3 \quad (59)$$

Vis-viva equation for Keplerian orbits:

$$v^2 = GM_{tot} \left(\frac{2}{r} - \frac{1}{a} \right) \quad (60)$$

Poisson's equation:

$$\nabla^2 \Phi = 4\pi G \rho \quad (61)$$

A few trig identities:

$$\cos(a+b) = \cos a \cos b - \sin a \sin b \quad (62)$$

$$\sin(a+b) = \sin a \cos b + \cos a \sin b \quad (63)$$

Some more calculus:

$$\frac{df}{dt} = \frac{\partial f}{\partial t} + \dot{\mathbf{v}} \cdot \nabla f \quad (64)$$

$$\ddot{x} = \frac{1}{2} \frac{d}{dx} \dot{x}^2 \quad (65)$$

Some integrals:

$$\int \frac{dx}{x+a} = \ln(x+a) + C \quad (66)$$

$$\int e^x dx = e^x + C \quad (67)$$

$$\int \ln x dx = x \ln x - x + C \quad (68)$$

$$\int \sin x dx = -\cos x + C \quad (69)$$

$$\int \cos x dx = \sin x + C \quad (70)$$

$$\int \frac{dx}{x^2+a^2} = \frac{1}{a} \tan^{-1} \left(\frac{x}{a} \right) \quad (71)$$

$$\int \frac{xdx}{\sqrt{x^2+a^2}} = \sqrt{a^2+x^2} \quad (72)$$

$$\int \frac{dx}{(a+x)^n} = \frac{(a+x)^{1-n}}{1-n} + C \quad (73)$$

$$\int \frac{xdx}{(x+a)^3} = \frac{a+2x}{2(a+x)^2} \quad (74)$$