-ggiagian Mechanics: (Backwards + Forwards)
for some variable of

$$
\begin{aligned}
& \left(m \frac{d}{d t}(\dot{r}) \frac{\partial r}{\partial q}\right) \delta q=-\frac{\partial V}{\partial q} \delta q \\
& \left.\left.m / \frac{d}{d t}\left[\left(\dot{r} \frac{\partial r}{\partial q}\right]-\dot{r}\right) \frac{d}{d t}\left(\frac{\partial r}{\partial q}\right)\right]=-\frac{\partial V}{\partial q} \quad\right) b \frac{d a}{d t}=\frac{d}{a t}(b a)-a \frac{d b}{d t} \\
& m\left(\frac{d}{d t}\left(\dot{r} \frac{\partial r}{\partial q}\right)-\dot{r} \frac{\partial \dot{r}}{\partial q}\right)=-\frac{\partial V}{\partial q} \\
& \left.\frac{\partial \dot{r}}{\partial \dot{q}}=\frac{1}{2} \frac{\partial}{\partial f}\left(\dot{r}^{2}\right) \quad\left[\frac{d r}{d t}=\frac{\partial r}{\partial t}+\dot{q} \frac{d r}{\partial q}\right]\right] \\
& \frac{d}{d t}\left[\frac{\partial}{\partial q}\left(\frac{1}{2} m^{2} r^{2}\right)\right]-\frac{\partial}{\partial q}\left(\frac{1}{2} m^{2}\right)=-\frac{\partial V}{\partial q}
\end{aligned}
$$

Define $T=\frac{1}{2} m r^{2}$

$$
\Rightarrow \frac{d}{d E}\left[\frac{\partial T}{\partial \dot{q}}\right]-\frac{\partial}{\partial q}(T-V)=0
$$

If $V \neq V(q)$, then

$$
\frac{d}{d t}\left[\frac{\partial(T-V)}{\partial \dot{q}}\right]-\frac{\partial}{\partial q}(T-V)=0
$$

Dope $L \equiv T-V$
Euler-Lagrage

$$
\Rightarrow \frac{d}{d t}\left(\frac{\partial L}{\partial q}\right)-\frac{\partial L}{\partial q}=0
$$ equation

Weird-looking equation, what does it mon?

$$
\begin{array}{ll} 
& \left(\frac{d}{d t}\left(\frac{\partial L}{\partial q}\right)-\frac{\partial L}{\partial q}\right)=0 \\
\Rightarrow & f(t)\left(\frac{d}{d t}\left(\frac{\partial L}{\partial q}\right)-\frac{\partial L}{\partial q}\right)=0 \\
\int_{t_{1}}^{t_{L}} f(t)\left(\frac{d}{d t}\left(\frac{\partial L}{\partial \dot{q}}\right)-\frac{\partial L}{\partial q}\right) d t=0 & (f(t) \times 0=0) \\
\int_{t_{1}}^{t_{2}} d t\left(\frac{d}{d t}\left(f(t) \frac{\partial L}{\partial q}\right)-\frac{\partial L}{\partial q} f(t)-f(t) \frac{\partial L}{\partial q}\right)=0 \\
\left.f(t) \frac{\partial L}{\partial q}\right|_{t_{1}} ^{t_{2}}-\int_{t_{1}}^{t_{1}} d t\left(\frac{\partial L}{\partial \dot{q}} \dot{f}+f \frac{\partial L}{\partial q}\right)=0
\end{array}
$$

If $f\left(t_{2}\right)=f\left(t_{1}\right)=0$, then $1^{\pi}$ term $=0$

$$
\begin{aligned}
& \Rightarrow \int_{t_{1}}^{t_{2}} d t\left[\frac{\partial L}{\partial \dot{q}} \frac{d f}{d t}+f\left(\frac{\partial L}{\partial q}\right)\right]=0 \\
& \left(\frac{\partial L}{\partial \dot{q}} \Delta \dot{q}\right) \frac{\partial L}{\partial q}(\Delta q) \\
& \Rightarrow \int_{t_{1}}^{t_{2}} d t\left[\frac{\partial L}{\partial \dot{q}} \Delta \dot{q}+\frac{\partial L}{\partial q} \Delta q\right]=0 \\
& \Rightarrow \int_{L_{1}}^{t_{2}} d t \Delta L(q, \dot{q}, t)=0 \\
& \Rightarrow \Delta\left\{\int_{t_{1}}^{t_{2}} d t L(q ; \dot{q}, t)\right\}=0
\end{aligned}
$$

$$
\Rightarrow \int_{t_{1}}^{t_{2}} d t\left[\frac{\partial L}{\partial \dot{q}} \Delta \dot{q}+\frac{\partial L}{\partial q} \Delta q\right]=0 \text { fer } \Delta q(t)=f(t)
$$

when $L \equiv T-V$
and verutions are done with
$\Rightarrow q(t), \dot{q}(t)$ are such that small variations of them pave SLat uncharged!

This is odd, what does it mem?
Take some system that is paramekirzed by $q(t)$. What does physics say the proper $g(t)$ is?


 "Action"

Somenhat cumbersome to consider all possible paths + integrate L, but we con starts with the culen-Lagrange equation.

This is exactly backwards from the tradifionol way of teaching this

Calculus of Variations
I. penal, we have on idea that the laws of plysts set ant some solution to a 1 problem, whe out could image nearby trajectories, the are "almost iphte How do we quantify this kind of thinkers? We need shore metrist de dine the cure co rightorwingr.Stat with a jerencel problem:

Take some notegical $\boldsymbol{S}=\int_{x_{1}}^{x_{2}} d x f\left(y(x), y^{\prime}(x)\right)$

$$
y^{\prime} \equiv \frac{d y}{d x}
$$

How do we minimizeS?
( H )

 greater $S$.
Note: freed at endpoints!

Write $y(x)=y_{0}(x)+\eta(x)$
host answer arbitrary af tot
If we wite $y(x)=y_{0}(x)+\alpha(x)$ ard tate $\alpha \rightarrow 0$, then $y(x)=y(x)$ for any $g(x)$

$$
\begin{gathered}
\Rightarrow \frac{\partial S}{\partial \alpha}=0 \text { at } y(x)=y_{0}(x) \\
S(\alpha)=\int_{x_{1}}^{x} f\left[y_{0}(x)+\alpha g(x), y^{\prime}\left(x+\alpha g^{\prime}(x), x\right] d x\right. \\
\frac{\partial S}{\partial \alpha}=\int_{x_{1}}^{x^{2}}\left(\frac{\partial f}{\partial y}+g^{\prime}(x) \frac{f f}{\partial y^{\prime}}\right) d x \quad g^{\prime}=\frac{d g}{d x} \\
=\frac{\partial f(y, j, x)}{\partial \alpha}
\end{gathered}
$$

Look of $z^{r^{b}}$ term: $\frac{d g}{d x} \frac{\partial f}{\partial y^{\prime}}=\frac{\alpha}{d x}\left(g \frac{\partial f}{\partial y^{\prime}}\right)-g \frac{d}{d x}\left(\frac{\partial f}{\partial y^{\prime}}\right)$

$$
\begin{aligned}
& \Rightarrow \frac{\partial S}{\partial \alpha}=\int_{x_{1}}^{x_{2}}\left(g(x) \frac{\partial f}{\partial y}-g \frac{d}{d x}\left(\frac{\partial f}{\partial y^{\prime}}\right)\right) d x+\underbrace{\int_{x}^{x_{2}} d x \frac{d}{d x}\left(g \frac{\partial f}{\partial y}\right)} \\
& \left.=g_{(x)}^{\partial y^{\prime}} \frac{f}{\partial y^{\prime}} \prime_{2}\right)-g\left(x_{1}\right) \frac{\partial f\left(y_{1}\right)}{\partial y^{\prime}} \\
& \text { If } g\left(x_{1}\right)=g\left(x_{2}\right)=0 \Rightarrow \text { |alt terms zero } \\
& \therefore \frac{\partial S}{\partial \alpha}=0 \Rightarrow \int_{x_{1}}^{x_{2}} d x\left[g(x) \frac{d f}{\partial y}-g(x) \frac{d}{d x}\left(\frac{\partial f}{\partial y}\right)\right]=0 \\
& \int d x g^{(x)}\left[\frac{\partial f}{\partial y}-\frac{d}{d x}\left(\frac{\partial f}{\partial y^{\prime}}\right)\right]=0
\end{aligned}
$$

But this has to be true for any $g(x)$ !

$$
\Rightarrow \frac{d}{d x}\left(\frac{\partial f}{\partial y^{\prime}}\right)-\frac{\partial f}{\partial y}=0
$$

"Eviler-Lagraye" equation

Brachistarnore Problem


How does one gut from $A$ to $B$ in minimum time usp just gravity?
Straight line? wat if $\qquad$

$$
T=\int \frac{d s}{v}
$$

$$
d s^{2}=d z^{2}+d l^{2}
$$

$$
v^{2}=2 g z
$$

$$
\begin{aligned}
\Rightarrow T & =\int \frac{\sqrt{d^{2}+d l^{2}}}{\sqrt{2 g z}} \\
& =\frac{1}{\sqrt{2 g}}\left(\frac{d z / \sqrt{1+\left(\frac{d y}{d z}\right)^{2}}}{\sqrt{z}}\right) \\
T & =\frac{1}{\sqrt{2 g}} \int_{A}^{B} d z f\left(y, l^{\prime}, z\right)
\end{aligned}
$$

$\begin{aligned} \text { Maize } T \Rightarrow \frac{d}{d z}\left(\frac{\partial f}{\partial l^{\prime}}\right) & -\frac{\partial f}{\partial l}=0 \\ & =0\end{aligned}$

$$
\begin{aligned}
& \Rightarrow \frac{\partial f}{\partial l^{\prime}}=\text { constant } \equiv c \\
& \Rightarrow \frac{1}{\not \partial} \frac{1}{\sqrt{z}} \frac{1}{\sqrt{1+l^{\prime 2}}} \cdot z l^{\prime}=c \\
& l^{\prime 2}\left(1-l^{\prime 2} z^{2}\right)=c^{2} z\left(1+l^{2}\right) \\
& l^{\prime 2}=\frac{c^{2} z^{2}}{1-c^{2} z^{2}} \\
& \Rightarrow \frac{d l}{d z}=\sqrt{\frac{c^{2} z}{1-c^{2} z}}
\end{aligned}
$$

$$
\begin{aligned}
& l=\sqrt{\frac{c^{2} z}{1-c^{2} z}} d z \\
& \frac{1 / \theta \int^{c^{2} z}}{\sqrt{1-c^{2} z}} \\
& \sin \theta=\sqrt{c^{2} z} \Rightarrow \sqrt{z}=\frac{\sin \theta}{\sqrt{c^{2}}} \\
& \cos \theta d \theta=\frac{\sqrt{c^{2}}}{2 \sqrt{z}} d z \\
& l=\int \frac{2 \tan \theta \cdot \cos \theta}{\sqrt{c^{2}}} \cdot \frac{\sin \theta}{\sqrt{c^{2}}} d \theta \\
& =\frac{2}{\sqrt{c^{4}}} \int \sin ^{2} \theta d \theta \\
& =\frac{1}{\sqrt{c^{4}}}\left(\theta-\sin ^{\theta} \theta \cos \theta\right) \\
& l=\frac{1}{c^{2}}\left[\theta-\frac{1}{2} \sin 2 \theta\right] \\
& =\frac{1 c^{2}}{2}[2 \theta-\sin 2 \theta] \\
& \sin \theta=\sqrt{c^{2} z} \\
& \Rightarrow \quad z=\frac{\sin \theta}{c^{2} \theta}=\frac{1}{2 c^{2}}(1-\cos 2 \theta) \\
& \Rightarrow(l, z)=\frac{1}{2 c^{2}}\left[\phi-\sin \phi, 1-\epsilon_{0}{ }^{3} \phi\right] \\
& \phi=28 \\
& =2 \sin ^{-1}\left(\sqrt{c^{2} z}\right) \\
& =R[\phi-\sin \phi, 1-\cos \phi]
\end{aligned}
$$

Wite

$$
\begin{aligned}
& R=\left(\frac{1}{2 c^{2}}\right) ; V=R \omega ; \phi=\omega t \\
& (l, z)=(V t-R \sin \omega t, R-R \cos \omega t)
\end{aligned}
$$

Define $L=T-V, T=\frac{1}{2} \dot{x}^{2}$

$$
\begin{array}{r}
V=V(\vec{x}) \\
\text { Not }+ \text {, so net total energy. }
\end{array}
$$

Why? Because it works.

$$
\begin{aligned}
& \frac{\partial L}{\partial x}=\frac{-\partial V}{\partial x}=F_{x} ; \quad ; \quad \frac{\partial L}{\partial y}=F_{y} ; \frac{\partial L}{\partial z}=F_{z} \\
& \frac{\partial L}{\partial \dot{x}}=m \dot{x} ; \quad \frac{\partial L}{\partial \dot{y}}=m \dot{y} ; \frac{\partial L}{\partial \dot{z}}=m \dot{z} \\
& \vec{f}=\frac{\vec{d}}{d t}=\frac{d}{d t}(m \dot{\vec{x}}) \\
& \Rightarrow \frac{\partial L}{\partial x}=\frac{d}{d t}\left(\frac{\partial L}{\partial \dot{x}}\right) ; \frac{\partial L}{\partial y}=\frac{d}{d t}\left(\frac{\partial L}{\partial \dot{y}}\right) ; \frac{\partial L}{\partial z}=\frac{d}{d t}\left(\frac{\partial L}{\partial \dot{z}}\right)
\end{aligned}
$$

equation for

$$
\text { equation for } S=\int L(\dot{x}, \dot{x}, t) d t
$$

Hamilton's Principle:

$$
\text { of } S=\int_{t_{1}}^{\frac{1}{2}} L d t
$$

Note that this
is nat Hamiltonian mechanics.
This ss usafinl beck es $\int_{x_{1}}^{x_{2}} L(\vec{x}, \vec{x}, t) d t$
along some path is a number. We are free to do the integral however we choose; for exande, we com do a charge of variables $\vec{x} \rightarrow \vec{q}$ with impunity, so lagrange equations must be varna in any coordinates.

In Catesren coords:

$$
\frac{\partial L}{\partial x}=F_{x}
$$

Byanalogy $\frac{d L}{\partial q} \equiv "$ genvalized Ferce"

$$
\frac{\partial L}{\partial \dot{q}} \equiv \text { "generalized Monnturn" }
$$

Why "genealized"? Do the same thing in prlan coordincites:


$$
\frac{\partial L}{\partial \dot{r}}=m \dot{r} \quad ; \frac{\partial L}{\partial \dot{\phi}}=\operatorname{mr}(\dot{r} \dot{\phi})
$$

To see that thrywork:

So, Lagrangien nechanics $\Leftrightarrow$ Newtorion mechanics

$$
\begin{aligned}
& \frac{\partial L}{\partial q}=\frac{d}{d t}\left(\frac{\partial L}{\partial q}\right) \quad \Leftrightarrow\left(-=\frac{d L}{d t}\right) \\
& \Rightarrow \quad f_{r}+\frac{m r^{2} \dot{q}}{r}=m r \\
& \left(F_{\phi} r\right)=\dot{m} \dot{\phi} r^{2}+2 m r(\ddot{r} \dot{\phi}) \\
& \Rightarrow F_{r}=\frac{m \dot{r}-m(r \dot{\phi})^{2}}{r} \\
& \Rightarrow F_{\phi}=m r \ddot{\phi}+2 m \dot{r} \dot{\phi}
\end{aligned}
$$

$$
\begin{aligned}
& L=T-4=\frac{1}{2} m\left(\dot{r}^{2}+r^{2} \dot{\phi}^{2}\right)-U\left(r_{i} \phi\right)
\end{aligned}
$$

torque

- Gefale Contrarts: Recal whenestated, I terved Hamilter's promiple statary from

$$
\begin{aligned}
& \stackrel{D}{F} \cdot d l=m \stackrel{*}{c} \cdot d l \\
& \text { 是 work-ancray. }
\end{aligned}
$$

If ve intreduce new ferces that down ark, than Lageonge's cquation must stil be vdid?
(The niotion wages beccesce the ane fener legnes of froedtim, trough)
Examples:
(1) Penduivm
$z \tilde{b}$

$$
\begin{aligned}
L & =T-u \\
& =\frac{1}{2} m l^{2} \phi^{2}-m g l(1-i \phi \phi)
\end{aligned}
$$

I degue of fuelon o
[2deges of $(x, y)$ renples I max thrk that woll do ant dave]

$$
\begin{aligned}
\frac{d}{\partial t}\left(\frac{\partial L}{\partial \dot{\phi}}\right) & =\frac{\partial L}{\partial \phi} \\
\frac{d}{d t}\left(m l^{2} \dot{\phi}\right) & =-m g \lambda \sin \phi \\
\ddot{\phi} & =-\left(\frac{g}{l}\right) \sin \phi
\end{aligned}
$$

$1_{0}^{z}$
Note that chaosng $z$ origin to pe at pivat wald give $-m g l \cos \phi=V$ (diffenat L)
In penenal there are differest fenctional forms of $L$ that waild waks

$$
S=\int_{t_{1}}^{t_{1}} L d t
$$

$S=\int_{t_{1}}^{L} d t$
take $L^{\prime}=L^{\prime}+\frac{d F}{1 L} \Rightarrow S=\int_{t_{1}}^{t}\left(L^{\prime}+\frac{d F}{d t}\right) d t=\int_{t_{1}}^{t_{2}} L^{\prime} d t+\underbrace{f\left(t_{2}\right)-f\left(t_{i}\right)}_{1})$
(2) Pondulim in acceleatty can


$$
\begin{aligned}
\vec{r} \equiv \vec{x} & =(x, z) \\
& =\left(\frac{1}{2} a t^{2}+l \sin \phi,-l \cos \phi\right) \\
\Rightarrow \vec{r} & =(a t+l \cos \phi \dot{\phi}, l \sin \phi \dot{\phi}) \\
\Rightarrow|\vec{r}|^{2} & =(a t+l \cos \phi \dot{\phi})^{2}+l^{2} \sin ^{2} \phi \dot{\phi}^{2} \\
& =a^{2} t^{2}+2 a t l \cos \phi \dot{\phi}+l^{2} \dot{\phi}^{2}\left(\cos ^{2} \phi+\sin ^{2} \dot{\phi}\right) \\
\therefore L & =T-V \\
& =\frac{1}{2} m\left(a^{2} t^{2}+2 a t l \cos \phi \dot{\phi}+l^{2} \dot{\phi}^{2}\right)+m g l \cos \phi
\end{aligned}
$$

conly deparls on $\phi, t, \dot{\phi}$

$$
\begin{aligned}
\frac{d}{d t}\left(\frac{\partial L}{\partial \dot{\phi}}\right) & =\frac{\partial L}{\partial \phi} \\
\operatorname{Lix} \frac{d}{d t}\left[h a t l \cos \phi+\lambda l^{2} \dot{\phi}\right] & =-a t \tan l \sin \phi \dot{\phi}-n g l \sin \phi \\
a l \cos \phi-a t l \sin \phi \dot{\phi}+l^{2} \dot{\phi} & =-\frac{1}{L} a t l \sin \phi \dot{\phi}-g l \sin \phi \\
l^{2} \dot{\phi} & =-g l \sin \phi
\end{aligned}
$$

What is eqin? $0=-g l \sin \phi-a x \cos \phi$

$$
\Rightarrow \tan \phi=-\frac{a}{g}
$$



Atwo od's Machine


$$
\begin{aligned}
& V=-m_{1} g x-m_{2} g(l-x) \\
& T=\frac{1}{2} m_{1} \dot{x}^{2}+\frac{1}{2} m_{2} \dot{x}^{2}
\end{aligned}
$$

$$
\begin{aligned}
\Rightarrow L & =\frac{1}{2}\left(m_{1}+m_{2}\right) \dot{x}^{2}+m_{1} g x+m_{2} g(l-x) \\
\frac{\partial L}{\partial x} & =m_{1} g-m_{2} g=\left(m_{1}-m_{2}\right) g \quad \leftarrow^{\prime \prime} f_{\text {ore" }} \\
\frac{\partial L}{\partial \dot{x}} & =\left(m_{1}+m_{2}\right) \dot{x} \\
\Rightarrow \quad \frac{d}{d t}\left[\left(m_{1}+m_{2}\right) \dot{x}\right] & =\left(m_{1}-m_{2}\right) g \\
\ddot{x} & =\frac{m_{1}-m_{2}}{m_{1}+m_{2}} g
\end{aligned}
$$



Rotating Coordinates
Free partide:

$$
\vec{\omega}=\omega \hat{z}
$$

$$
L=\frac{1}{2} m \dot{\vec{r}}^{2}
$$

inertiol coordinates
Rotating coodmates:

$$
\begin{aligned}
& \left.\begin{array}{l}
x^{\prime}=x \cos \omega t+y \sin \omega t \\
y^{\prime}=y \cos \omega t-x \sin \omega t \\
z^{\prime}=z
\end{array}\right\} \Rightarrow \begin{array}{l}
x=x \cos \omega t-y \sin \omega t \\
y=y \cos \omega t+x \sin \omega t
\end{array} \\
& \dot{x}=\dot{x}^{\prime} \cos t-\dot{y}^{\prime} \sin n t-x^{\prime} \cos \sin t-y^{\prime} \cos ^{\prime} \tan \theta \\
& \dot{y}=\dot{y}^{\prime} \cos \omega t+\dot{x}^{\prime} \sin t-y^{\prime} \omega \sin n t+x w \cos a t \\
& \dot{x}=\cos \omega t\left(\dot{x}^{\prime}-\omega y^{\prime}\right)-\sin t\left(\dot{y}^{\prime}+x^{\prime} \omega\right) \\
& \dot{y}=\cos \omega t\left(\dot{y}^{\prime \prime}+\omega x^{\prime}\right)+\sin \omega t\left(\dot{x}^{\prime}-y^{\prime} \omega\right) \\
& \ddot{x}^{2}+\dot{y}^{2}=\left(\dot{x}^{\prime}-w y^{\prime}\right)^{2}+\left(\dot{y}^{\prime}+x^{\prime} w\right)^{2} \\
& \Rightarrow L=\frac{1}{2} m\left\{\left(\dot{x}^{\prime}-\omega y^{\prime}\right)^{2}+\left(y^{\prime}+\omega x^{\prime}\right)^{2}+z^{2}\right\} \\
& \frac{d}{d t}\left(\frac{\partial L}{\partial \dot{x}^{\prime}}\right)=\frac{\mu L}{\partial x^{\prime}} \Rightarrow \frac{d}{d t}\left[m\left(\dot{x}^{\prime}-w y^{\prime}\right)\right]=m\left(\dot{y}^{\prime}+w x\right) w \\
& \ddot{x}=\omega^{2} x+2 \omega\left(\dot{y}^{\prime}\right) \\
& \frac{d}{d t}\left(\frac{d L}{\partial \dot{y}}\right)=\frac{\partial L}{\partial y^{\prime}} \Rightarrow \quad \quad^{\prime}=\omega^{2} y-2 \omega^{\prime} \dot{x} \\
& \Rightarrow(\dot{x}, \ddot{y})=\omega_{\text {Centrifygal }}^{\omega^{2} \vec{R}}-\underbrace{2 \vec{w} \times \stackrel{\rightharpoonup}{\vec{R}}}_{\text {Coridtis }} \text { [Jus ithe a } B \text { freld] }
\end{aligned}
$$

Spharical Pendulum


$$
\left\{\begin{array}{l}
x=l \cos \psi \sin \theta \\
y=l \sin \phi \sin \theta \\
z=-l \cos \theta
\end{array}\right.
$$

$$
L=\frac{1}{2} m\left(\dot{x}^{2}+\dot{y}^{2}+\dot{z}^{2}\right)+m g l \cos \theta
$$

$$
=\frac{1}{2} m l^{2}\left\{\begin{array}{l}
\sin ^{2} \theta \sin ^{2} \phi \dot{\phi}^{2}+\cos ^{2} \phi \cos ^{2} \theta \dot{\theta}^{2} \\
+\cos ^{2} \phi \sin ^{2} \theta \dot{\phi}^{2}+\sin ^{2} \phi \cos ^{2} \phi \dot{\theta}^{2}
\end{array}\right.
$$

$$
\begin{aligned}
& \left.+\sin ^{2} \theta \theta^{2}\right\}^{2}+m \\
& \left(\dot{\phi}^{2} \sin ^{2} \theta+\dot{\theta}^{2}\right)+m g l \cos \theta
\end{aligned}
$$

$$
\frac{d}{d t}\left(\frac{\partial L}{\partial \phi}\right)=\frac{1 L}{\partial \phi} \Rightarrow m l^{2} \sin ^{2} \theta \ddot{\phi}=0 \Rightarrow \quad \Rightarrow \quad \begin{aligned}
& l^{2} \sin ^{2} \theta \dot{\phi}=\text { constat }=J_{\phi} \\
& \\
& \text { Corsmation of aijular }
\end{aligned}
$$ monetion)

$$
\begin{aligned}
& d /(\alpha / \dot{\partial})=\partial / \partial \theta \Rightarrow l^{2} \ddot{\theta}=m l^{2} \dot{\phi}^{2} \sin \theta \cos \theta-m g l \sin \theta \\
& \Rightarrow \begin{aligned}
& m l^{2} \ddot{\theta}= m l^{2} J_{\phi}^{2} \sin ^{\theta} \cos \theta-m g l \sin \theta \\
&\left(m l^{2}\right)^{2} \sin ^{4} \theta
\end{aligned} \\
& \Rightarrow \dot{\theta}=\frac{\frac{\partial}{q}{ }^{2} \cos \theta}{m l^{2} \sin ^{2} \theta}-\frac{g}{l} \sin \theta
\end{aligned}
$$

Foucault's Pendulum
for ease, let's go to north pole:


$$
z^{\uparrow}
$$

$$
z=-l \cos \theta
$$

$$
L=\frac{1}{2} m\left\{\left(\dot{x}^{\prime}-w y^{\prime}\right)^{2}+\left(y^{\prime}+w x^{\prime}\right)^{2}\right\}+1 \cos \theta m g
$$

$$
x^{2}+y^{2}=l^{2} \sin ^{2} \theta
$$

$$
=l^{2}\left(1-\cos ^{2} \theta\right)
$$

$$
\begin{aligned}
& \Rightarrow \cos ^{2} \theta=1-\left(\frac{x^{2}+y^{2}}{e^{2}}\right) \\
& \left.\Rightarrow \cos \theta \simeq \sqrt{1-\frac{\left(x^{2}+y^{2}\right.}{l^{2}}}\right) \simeq 1-\frac{1}{2}\left(\frac{x^{2}+y^{2}}{l^{2}}\right)
\end{aligned}
$$

$\Rightarrow L \simeq \frac{1}{2} m\{$

$$
\}+\ln g\left(1-\frac{1}{2} \frac{x^{2}+y^{2}}{l^{2}}\right)
$$

$$
\begin{equation*}
\frac{d}{d t}\left(\frac{\partial L}{\partial x}\right)=\frac{\partial L}{\partial x} \tag{2}
\end{equation*}
$$

$$
\begin{aligned}
& \ddot{x}^{\prime}=\omega^{2} x+2 \omega \dot{y}^{\prime}-x\left(\frac{g}{l}\right) \\
& \ddot{y}^{\prime}=\omega^{2} y-2 \omega \dot{x}^{\prime}-y\binom{g}{l}
\end{aligned}
$$

$\Rightarrow$ Regular pendulum with systematic deflection to the right as it oscillates,

Falling Stick


Physically, no force in $x$-direction so $\dot{x}_{c}=0$, but left's imagine we don't know that

$$
\begin{aligned}
& \frac{d}{d t}\left(\frac{\partial L}{\partial \dot{x}_{6 m}}\right)=\frac{\partial L}{\partial \dot{x}_{c o m}} \\
& \Rightarrow m \ddot{x}_{\text {com }}=0 \\
& \frac{d}{d t}\left(\frac{\partial L}{\partial \dot{\phi}}\right)=\frac{d L}{\partial \phi} \\
& \frac{d}{d t}\left(\frac{m L^{2}}{4} \cos ^{2} \phi \dot{\phi}+I \dot{\phi}\right)=-\frac{m q}{2} \cos \phi-\frac{m L^{2}}{4} \cdot \cos \phi \sin \phi \dot{\phi}^{2} \\
& \left(\frac{m L^{2} \cos ^{2} \phi}{4}+I\right)^{\prime} \dot{\phi}=\frac{m L^{2}}{4} \cdot \cos \phi \sin \phi \dot{\phi}^{2}-\frac{m g L}{2} \cos \phi \int_{\substack{\text { But This } \\
\text { Mnswe }}}^{\substack{\text { Mess } y}}
\end{aligned}
$$


$\xrightarrow[\text { FORCES }]{\text { CoNSTENWI }}$

$$
\begin{aligned}
& L=\frac{1}{m \theta^{2} \omega^{2}+\frac{1}{2} m R^{2} \theta^{2}-m g R(1-\cos \theta)} \\
& \theta \frac{\left(T^{2} m R^{2} \sin ^{2} \theta \omega^{2}\right.}{2}+\frac{2}{L}=\frac{L}{L(\theta, \theta)} \\
& \frac{d}{d t}\left(\frac{\partial L}{\partial \dot{\theta}}\right)=\frac{\partial L}{\partial \theta} \\
& m R^{2} \dot{\theta}=m R^{2} \omega^{2} \sin \theta \cos \theta-m g R \sin \theta \\
& \Rightarrow \ddot{\theta}=\left(\omega^{2} \cos \theta-\frac{g}{R}\right) \sin \theta
\end{aligned}
$$

Nates: (1) equitibrm when $\sin \theta=0$ OR $\omega^{2} \cos \theta=g_{R}$
(2) $\cos \theta=\frac{g}{\beta^{2} \omega^{2}}$ is a funny spot : as $\omega \rightarrow 0, \cos \theta \rightarrow \infty$

But $\cos ^{\theta}+\mathrm{H}^{\prime} \Rightarrow$ unghycial at low W
$\Rightarrow$ solution pops into existence vimen $R \omega^{2}=g$

$$
(1 . e \text { at } \theta=0)
$$

(3) We never tolled about the wire!

* we sack it in by assenting that this bead was rotating with $v=\rho \omega$
So, we put the contract in by hand without thintore '

In gennal how do we hande contraunts?


Block on a weage.

$$
L=\frac{1}{2} m\left(\dot{x}^{2}+\dot{z}^{2}\right)-m g y
$$

$$
\begin{equation*}
y=y_{0}-\left(\frac{y_{0}}{x_{0}}\right) x \tag{1}
\end{equation*}
$$

$$
\begin{gathered}
\Rightarrow \dot{y}=-\frac{y_{0}}{x_{0}} \dot{x} \\
\Rightarrow L=\frac{1}{2} m\left(\dot{x}^{2}+\left(\frac{y_{0}}{x_{0}}\right)^{2} \dot{x}^{2}\right)+m g\left(\frac{y_{0}}{x_{0}}\right) \times \text { rgg }
\end{gathered}
$$

$$
\begin{equation*}
L=\frac{1}{2} m \dot{x}^{2}\left(1+\frac{y_{0}^{2}}{x_{0}^{2}}\right)+m g\left(\frac{y_{0}}{x_{0}}\right) x \tag{2}
\end{equation*}
$$

$$
\frac{d}{d t}\left(\frac{\partial L}{\partial \dot{x}}\right)=\frac{\partial L}{\partial x}
$$

$$
\left(1+\frac{y_{0}^{2}}{x_{0}^{2}}\right) m \dot{x}=m g \frac{y_{0}}{x_{0}}
$$

$$
\begin{aligned}
&\left.\ddot{x}=\frac{g\left(y \tilde{x}_{0}\right.}{\dot{x}}\right) \\
& 1+y_{0}^{2} x_{0}^{2}=g \frac{y_{0} \cdot x_{0}}{x_{0}^{2}+y_{0}^{2}} \\
&=(g \sin \theta) \cos \theta
\end{aligned}
$$

$$
=(q \sin \theta) \cos \theta
$$


(1) $\Rightarrow$

$$
\Rightarrow \ddot{y}=-\frac{y_{0}}{x_{0}} \ddot{x} \Rightarrow \ddot{y}=(g \sin \theta) \sin \theta
$$



* We hove acoedy picced oat porths that are consintet with constraits
* while $L=L(x, \dot{x})$, it is not als possible $[x(t), y(t)]$

Fir ok to just stick the constraints ito the Lagragion:
If $y(x)=f(x)$ is a requirement, ken just put $y=f(x)$ in for $y(x), \dot{y}(x)$ and do your stuff
Is thine a formal way to call out the constraints?
YES: Lagrage multipliers:
We wat to find paths that have $\iint_{t_{1}}^{t_{2}^{2}}(x, y, \dot{x}, y, t)=0 \quad$ for $[x(t), y(t)]$
But aloe satisfy $f(x, y)=0$
e.g. $y=y_{0}-\frac{y_{0}}{x_{0}} x \rightarrow \frac{y+\frac{x_{0}}{y_{0}} x-y_{0}}{f(x, y)}=0$


$$
\begin{aligned}
& f(x, y)=0 \\
& \Rightarrow \delta f(x, y)=0 \\
& \Rightarrow\left(\int X(t) d t\right) \delta f=0 \\
& \int \lambda(t) d t\left(\frac{\partial f}{\partial x} \delta x+\frac{\partial f}{\partial y} \delta y\right)=0
\end{aligned}
$$

So we can ADD thistern to the

$$
\begin{aligned}
& \delta S=0: \\
& \delta S=\left(\left(\frac{\partial L}{\partial x}+\frac{\left.\left.\lambda(t) \frac{\partial f}{\partial x}-\frac{d}{d t}\left(\frac{\partial L}{\partial \dot{x}}\right)\right)\right) \frac{\delta x}{\partial} d t}{}+\int\left(\frac{\partial L}{\partial y}+\lambda(t) \frac{\partial f}{\partial y}-\frac{d}{d t}\left(\frac{\partial L}{\partial \dot{y}}\right)\right) \delta y d t=0\right.\right.
\end{aligned}
$$

But now $\delta x$ ad $\delta y$ ae not completely independent, so this wont be true for Any $\lambda(t)$. However, therels sine friction $\lambda(t)$ for which this works:

$$
\begin{gathered}
\frac{\partial L}{\partial t}+\lambda(t) \frac{\partial f}{\partial f}-\frac{d}{d t}\left(\frac{\partial L}{\partial \dot{x}}\right)=0 \\
\frac{\partial L}{\partial y}+\lambda(t) \frac{\partial f}{\partial y}-\frac{d}{d t}\left(\frac{\partial L}{\partial y}\right)=0 \\
f(x, y)=0
\end{gathered}
$$

Now we have 3 equations a 3 unkeons $(x, y, \lambda)$

Seems kind of cryptic. How about a new try?
We are fold that some 3-D function $f(x, y, z)$ is constant along same surface. [surface of $c(x, y, z)=0$ ]

It must be the case that $\vec{\nabla} f$ is II to $\hat{h}$

$$
\Rightarrow \vec{v} f=\alpha \hat{n}
$$

What's $n$ ? Prepentional to the grodeat of haw the swifface is defined.
If $C(x, y, z)=0$ defines the surface,
then $\vec{v} f=w(x, y, z) \vec{\nabla} C$ when $C=0$
At $C=0$, we can wite this as

$$
\vec{v} f=\vec{q}(w \cdot c)
$$

for $\lambda=-\omega$

$$
\Rightarrow \quad \vec{v}(f+\lambda c)=0 \text { when } C=0
$$

Now we have a recipe for firming $f(x, y, z)$ such that $f=$ constant on a surface where $C=0$.
How dos this relate to mechanics?
Notice that if the are 2 constants: $f=$ Constant the $C_{1}=0$
AND $C_{2}=0$
then we con just add the to the nix

$$
\vec{\nabla}\left(f+\lambda_{1} C_{1}+\lambda_{2} C_{2}\right)=0 \quad \begin{gathered}
\text { (must be normal to } \\
\text { Bert surfaces) }
\end{gathered}
$$

* For constraints in mechanics, $C_{1}$ would be the requinmant that at sone time $t_{1}$ the objet has to be confined to some region of space. cg. on a table at height $z=z_{0} \Rightarrow \quad z\left(t_{1}\right)-z_{0}=0$
\& At tine $t_{2}$ this thing hastasent $z z z_{0} \Rightarrow z\left(t_{2}\right)-z=0$
$\Rightarrow$ there are an infinite number of constraints $C_{i}=0$
that are of the form $z\left(t_{i}\right)-z_{0}=0$
* We also know that the action is stationary with respect to deviations firm the true path $\Rightarrow S=$ constant along the path.

$$
\begin{aligned}
\Rightarrow & \operatorname{sum} f \text { undradt }=\sum_{i=1}^{\infty} \lambda(t) C(t) \rightarrow \int d t \lambda(t) C(t) \\
\Rightarrow \delta & \left.\delta S+\int d t, \lambda(t) C(t)\right]=0 \\
& \delta\left[\int[L+\lambda(t) C(t)] d t\right]=0
\end{aligned}
$$

$\Rightarrow$ to include constant just add it to the Lagrangion with an arbitrary pre-factor!

One more way to thru about it:
For constraint $C(t)=0 \quad\left(\right.$ lite $\left.z(t)-z_{0}=0\right)$

$$
\begin{aligned}
S_{0} & =\int d t L \\
S & =\int d t(L+0) \\
& =\int d t(L+0 d(t)) \\
S & =\int d t[L+C(t) \lambda(t)]
\end{aligned}
$$

$\Rightarrow$ minimize $S_{0}$, minimize $S$
So far $\lambda(t)$ is arbitrary, but the is a consistency refuicenat $:(t)=0$ fr a given $d(t)$,
c.g. Pendulum
take coordinates $[r(t), \theta(t)]$

$$
L=\frac{1}{2} m\left(r^{2}+r^{-2} \theta^{2}\right)-m g(1-\cos \theta)
$$

But now we have a constant: $r(t)-R=0$

$$
\begin{aligned}
\Rightarrow L & =\frac{1}{2} m\left(\dot{r}^{2}+r^{2} \dot{\theta}^{2}\right)-m g r(1-\cos \theta)+\lambda(t)[r(t)-R] \\
\frac{d}{d t}\left(\frac{\partial L}{\partial \dot{f}}\right) & =\frac{\partial L}{\partial q} \\
\Rightarrow m^{*} \dot{r} & =m r \dot{\theta}^{2}+m g \cos \theta+\lambda(t)
\end{aligned}
$$

But we also have $r(t)-R=0 \Rightarrow \ddot{r}=0$

$$
\Rightarrow \lambda(t)=-m r \theta^{2}-m g \cos \theta
$$

What is $\lambda(t)$ ?

$$
\begin{aligned}
a_{r} & =\frac{v^{2}}{r}=r \dot{\theta}^{2} \\
a & =\frac{F}{m}=\frac{I}{m}-\frac{m g}{m} \cos \theta \\
\Rightarrow T & =m r \dot{\theta}^{2}+m g \cos \theta \\
\Rightarrow \lambda(t) & =-T \quad Z
\end{aligned}
$$

Whoa
The constant force pops out as berg $\lambda(t)$ in this case. In general, recall "generalized force"

$$
f_{i}=\frac{\partial L}{\partial q_{i}}
$$

no recall that we added a tern to the Lagranion

$$
\Rightarrow \frac{L_{c}=\lambda(t) \cdot c(t)}{F_{(i)}=\frac{\partial L_{c}}{\partial q_{i}}=\lambda(t) \frac{\partial c\left(\vec{q}_{i} t\right)}{\partial q_{i}}}
$$

So you have 2 choices vien dealig woth corotrants

$$
C(x, y, z, t)=\theta:
$$

(1) we the cinstrants to elininate an of the cooctrates e.g. write $y(x)=\cdots ; \dot{y}(x)=\ldots$ ad then ignore $y$ to solve.
(2) bep al coordmates and add $\lambda(t) C(t)$ to the lagrazion, inforcing $C(t)=0$
eg. Blok ch a wedge:
Constant: $\quad y+\frac{y_{0}}{x_{0}} x-y_{0}=0$


$$
\begin{aligned}
& \text { x: } \quad \frac{d}{d E}\left(\frac{\partial L}{\partial \dot{x}}\right)=\frac{\partial L}{\partial x} \Rightarrow m^{\ddot{x}}=\lambda(t) \frac{y_{0}}{x_{0}} \\
& \text { y: } \frac{d}{d t}\left(\frac{\partial L}{\partial \dot{y}}\right)=\frac{\partial L}{\partial y} \Rightarrow{ }^{m \dot{y}}=-m g+\lambda(t) \\
& \int_{\lambda(t)}=m \ddot{y}+m y \\
& \Rightarrow m^{\dot{x}}=\frac{y_{0}}{x_{0}}\left(m^{\dot{y}}+m y\right)
\end{aligned}
$$

Now use contrant: $\dot{y}=-\frac{y_{0}}{x_{0}} \vec{x}$

$$
\begin{aligned}
& \Rightarrow \ddot{x}=\frac{y_{0}}{x_{0}}\left(m g-\frac{y_{0}}{x_{0}} \ddot{x}\right) \\
& \Rightarrow \ddot{x}\left(1+\frac{y_{0}^{2}}{x_{0}^{2}}\right)=\frac{y_{0}}{x_{0} g} \\
& \left.\Rightarrow \ddot{x}=\frac{g\left(y_{0} / x_{0}\right)}{1+y_{0}^{2} / x_{0}^{2}} \quad \text { Ca befare }\right)
\end{aligned}
$$



AND: $\lambda(t)=x_{0} \sin _{0} \dot{x}_{0}=\frac{m g}{1+y_{i}^{2} x_{0}^{2}}=(m g \cos \theta) \cos =N_{y}$

