Damped / Driven Linear Oscillators

EOM : $\ddot{x} + 2\beta\dot{x} + \omega_0^2 x = f(t)$ where $\vec{F}_{\text{restore}} = -k\vec{x} = -m\omega_0^2\vec{x}$, $\vec{F}_{\text{damp}} = -b\vec{v} = -m(2\beta)\vec{v}$, $\vec{F}_{\text{drive}} = m f(t)$

Damping Cases : Critical damping : $\beta = \omega_0 \equiv \sqrt{\frac{k}{m}}$, Underdamping: $\beta < \omega_0$, Overdamping: $\beta > \omega_0$

Resonance:

Quality factor $Q = \dfrac{\omega_{\text{peak}}}{\Delta\omega_{\text{FWHM}}}$ where $A^2\left(\omega_{\text{peak}} \pm \dfrac{\Delta\omega_{\text{FWHM}}}{2}\right) = \dfrac{1}{2}A^2(\omega_{\text{peak}})$ and $A \equiv$ amplitude of $x(t)$

$Q = 2\pi \dfrac{E_{\text{stored}}}{E_{\text{dissipated / cycle}}}$ where $E_{\text{stored}} = T + U$ for mechanical oscillator

For weak damping $\beta \ll \omega_0$: $\omega_{\text{peak}} \approx \omega_0$ $A_{\text{peak}} \approx \dfrac{f_0}{2\beta\omega_0}$ $Q \approx \dfrac{\omega_0}{2\beta}$

Coupled Linear Oscillators

$$\mathbf{M}\ddot{\vec{q}} = -\mathbf{K}\vec{q} \qquad T = \frac{1}{2}\mathbf{M}_{ij}\dot{q}_i\dot{q}_j \qquad U = \frac{1}{2}\mathbf{K}_{ij}q_i q_j \qquad \mathbf{M}_{ij} = \left.\frac{\partial^2 T}{\partial\dot{q}_i\partial\dot{q}_j}\right|_{\vec{q}=0} \qquad \mathbf{K}_{ij} = \left.\frac{\partial^2 U}{\partial q_i\partial q_j}\right|_{\vec{q}=0}$$

Transverse oscillations of taut string: $k_T = \left.\dfrac{\text{tension}}{\text{length}}\right|_{\text{equilib}}$

Fourier Series as a Linear Vector Space

- **Space** : $|f\rangle \equiv$ **τ-periodic functions** $f(t)$ that are periodic over $t = [-\tau/2 \to \tau/2]$, with $\omega = 2\pi/\tau$
  - **Basis #1** : $|n\rangle \equiv \begin{cases} \sin(n\omega t) & n = 1,\ldots,\infty \\ 1/\sqrt{2} & n = 0 \\ \cos(n\omega t) & n = -1,\ldots,-\infty \end{cases}$
  - **Inner Product #1** : $\langle g|f\rangle \equiv \dfrac{2}{\tau}\displaystyle\int_{-\tau/2}^{\tau/2} g(t)f(t)\,dt$
  - **Basis #2** : $|n\rangle = e^{in\omega t}$
  - **Inner Product #2** : $\langle \tilde{g}|\tilde{f}\rangle \equiv \dfrac{1}{\tau}\displaystyle\int_{-\tau/2}^{\tau/2} g^*(t)f(t)\,dt$

➔ **Basis is Orthonormal** : $\langle n|m\rangle = \delta_{nm}$

➔ **Completeness** : any $|f\rangle = \displaystyle\sum_{n=-\infty}^{+\infty} |n\rangle\langle n|f\rangle$

Lagrangian Mechanics

* Gen. coord q_i must be indep

$S \equiv \displaystyle\int_{t_1,\vec{q}_1}^{t_1,\vec{q}_1} dt\, L(q_i, \dot{q}_i, t)$ $\delta S = 0 \to \dfrac{\partial L}{\partial q_i} = \dfrac{d}{dt}\left(\dfrac{\partial L}{\partial \dot{q}_i}\right)$ for each q_i

$H \equiv \dot{q}_i(\partial L/\partial\dot{q}_i) - L$ conserved when $\partial L/dt = 0$

Principle of Least Action :

$L = T - U \;\to\; \delta S = 0$ @ true $\{q_i(t)\}$

Gen. force $Q_i \equiv \dfrac{\partial L}{\partial q_i}$, momentum $p_i \equiv \dfrac{\partial L}{\partial \dot{q}_i}$

$H \equiv p_i\dot{q}_i - L$ equals T+U when $\vec{r}_a = \vec{r}_a(q_i)$