2-Body Central Force Problems & Scattering

- **Coordinates & Reduced Mass** : $\vec{r}_1 = \vec{R} + \frac{m_2}{M}\vec{r}$, $\vec{r}_2 = \vec{R} - \frac{m_1}{M}\vec{r}$, $\vec{r} \equiv \vec{r}_1 - \vec{r}_2$, $\mu = \frac{m_1 m_2}{M}$
- **Centrifugal** force & PE : $\vec{F}_{cf} = \frac{L^2}{\mu r^3}\hat{r}$, $U_{cf} = \frac{L^2}{2\mu r^2}$, effective radial $U^* = U + U_{\text{cf}}$
- **Angular EOM** : $\dot{\phi} = \frac{L}{\mu r^2}$ • **Radial EOMs** : $\mu\ddot{r} = F(r) + F_{\text{cf}}(r)$, $E = T + U = \frac{1}{2}\mu\dot{r}^2 + U_{\text{cf}} + U$
- **Path Equation** : $u(\theta) \equiv 1/r(\theta) \rightarrow u'' + u = -\frac{\mu F(1/u)}{L^2 u^2}$ and $u' = -\frac{\mu\dot{r}}{L}$
- **Conics** : With (r, ϕ) centered on a focal point and E ≡ Ellipse, H ≡ Hyperbola

$$\frac{1}{r} = \frac{a}{b^2}(\pm 1 + e\cos\phi) \text{ with } \begin{cases} +\text{: E or H-near-branch} \\ -\text{: H-far-branch} \end{cases}, \quad e = \frac{c}{a} = \frac{\sqrt{a^2 \mp b^2}}{a} \text{ with } \begin{cases} -\text{: E} \\ +\text{: H} \end{cases}$$

- **Kepler Orbits** $F = -\frac{\gamma}{r^2}$: $r(\phi) = \frac{r_0}{\text{sgn}[\gamma] + e\cos\phi}$ with $r_0 = \frac{L^2}{\mu|\gamma|} = \frac{b^2}{a} = a\left|1 - e^2\right|$, $E = \mp\frac{|\gamma|}{2a} = \frac{|\gamma|(e^2 - 1)}{2r_0}$

  Bounded orbits: $\tau^2 = \frac{4\pi^2\mu}{\gamma}a^3$, $r_0 = \frac{2r_{\min}r_{\max}}{r_{\min} + r_{\max}}$, $e = \frac{r_{\max} - r_{\min}}{r_{\max} + r_{\min}}$

  Unbounded orbits: scattering angle $\theta = \pi - 2\alpha$ with $\tan\alpha = \frac{b}{a}$, impact parameter $b_0 = b$ ☺

- **XSec** : $d\Omega \equiv \frac{dA}{r^2} = \begin{cases} \sin\theta\, d\theta\, d\phi \\ d\theta_x\, d\theta_y \end{cases}$, $\frac{d\sigma}{d\Omega} = \frac{b}{\sin\theta}\left|\frac{db}{d\theta}\right|$ with $\theta =$ scattering angle • **Lumi**: $\mathcal{L} = n_A N_e$, $\frac{dN_{\text{ev}}}{dt} = \mathcal{L}\sigma$

- **Earth Specifications**: $R_\oplus = 6.4 \times 10^6$ m, $GM_\oplus = gR_\oplus^2$, $g = 10$ m/s^2
- **Earth Orbit**: $r_0 = a = 1$ A.U. $= 1.5 \times 10^{11}$ m, orbital velocity $v_\oplus = 3 \times 10^4$ m/s
- **Atomic Data**: 1 amu ≈ mass of 1 nucleon (proton or neutron) = 1.66×10^{-27} kg = $(5/3) \times 10^{-24}$ g
  Gas at STP has $N_{\text{Avog}} = 6.02 \times 10^{23}$ molecules in 22.4 liters; 1 **barn** = 10^{-28} m^2 = 10^{-24} cm^2

Lagrangian Mechanics

* Gen. coord q_i must be indep

$$S \equiv \int_{t_1, \vec{q}_1}^{t_1, \vec{q}_1} dt\, L(q_i, \dot{q}_i, t) \qquad \delta S = 0 \rightarrow \frac{\partial L}{\partial q_i} = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) \text{ for each } q_i$$

$H \equiv \dot{q}_i(\partial L/\partial \dot{q}_i) - L$ conserved when $\partial L/dt = 0$

Principle of Least Action :

$L = T - U \rightarrow \delta S = 0$ @ true $\{q_i(t)\}$

Gen. force $Q_i \equiv \frac{\partial L}{\partial q_i}$, momentum $p_i \equiv \frac{\partial L}{\partial \dot{q}_i}$

$H \equiv p_i\dot{q}_i - L$ equals T+U when $\vec{r}_a = \vec{r}_a(q_i)$