System = mass in uniform gravity, 1D vertical motion
Endpoints = (y,t) from $(0,-\tau)\to(0,+\tau)$
Lagrangian $L(y,\dot{y},t)\,\frac{2}{m}=\dot{y}^2-2gy$

Path Class #1 : **constant speed** going up & coming down → vary **H** = max height

- **Paths** $\boxed{y(t)=H\left(1-\frac{|t|}{\tau}\right)}$ → $\dot{y}=\pm\frac{H}{\tau}$

- **Lagrangian** $L(t)\,\frac{2}{m}=\left(\frac{H}{\tau}\right)^2-2gH\left(1-\frac{|t|}{\tau}\right)$ → $L(t)\,\frac{2}{m(g\tau)^2}=\left(\frac{H}{g\tau^2}\right)^2-2\frac{H}{g\tau^2}\left(1-\frac{|t|}{\tau}\right)$

  Define $b\equiv\frac{H}{g\tau^2}$ and vary that ... true path has $H=\frac{g\tau^2}{2}$ ∴ $b_{\text{true}}=\frac{1}{2}$

  Drop $\frac{2}{m(g\tau)^2}$ → $\boxed{L(t)=b^2-2b\left(1-\frac{|t|}{\tau}\right)}$ … vary b and see what happens !

```
plot b^2-2b(1-Abs[t]) for b={0.1,0.25,0.5,0.75,1} from t=-1 to 1
```

- **Action** $\boxed{S=\int_{-\tau}^{+\tau}L\,dt=2\tau\,b\,(b-1)}$

```
plot b(b-1) from b=0 to 1.5
```

- **Least Action** $\frac{1}{2\tau}\left.\frac{dS}{db}\right|_{b^*}=0=2b^*-1$ → min action at $\left(b^*=\frac{1}{2}\right)=b_{\text{true}}$ ☺ → $\left(S_{\min}=-\frac{\tau}{2}\right)$

Path Class #2 : **constant acceleration** up & down → vary **a** = acceleration

- **Paths** $y(t)=v_0(t+\tau)-\frac{1}{2}a(t+\tau)^2$ and $y(\pm\tau)=0$ ∴ $v_0=a\tau$ → $\boxed{y(t)=\frac{a}{2}\left(\tau^2-t^2\right)}$

- **Lagrangian** $L(t)\,\frac{2}{m}=a^2t^2+ga\left(t^2-\tau^2\right)$ → $L(t)\,\frac{2}{m(g\tau)^2}=\frac{a}{g}\left[\frac{t^2}{\tau^2}\left(\frac{a}{g}+1\right)-1\right]$

  Define $b\equiv\frac{a}{g}$ and vary that ... true path has $a=g$ ∴ $b_{\text{true}}=1$

  Drop $\frac{2}{m(g\tau)^2}$ → $\boxed{L(t)=b\left[\frac{t^2}{\tau^2}(b+1)-1\right]}$ … vary b and see what happens !

```
plot b(t^2(b+1)-1) for b={0.4,0.7,1,1.3,1.6} from t=-1 to 1
```

- **Action** $\boxed{S=\int_{-\tau}^{+\tau}L\,dt=\frac{2\tau b}{3}(b-2)}$

```
plot b(b-2) from b=0 to 1.5
```

- **Least Action** $\frac{3}{2\tau}\left.\frac{dS}{db}\right|_{b^*}=0=2b^*-2$ → min action at $\left(b^*=1\right)=b_{\text{true}}$ → $\left(S_{\min}=-\frac{2\tau}{3}\right)<-\frac{\tau}{2}$ ☺