Coupled Linear Oscillators

$$\mathbf{M}\ddot{\vec{q}} = -\mathbf{K}\vec{q} \qquad T = \frac{1}{2}\mathbf{M}_{ij}\dot{q}_i\dot{q}_j \qquad U = \frac{1}{2}\mathbf{K}_{ij}q_i q_j$$

$$\mathbf{M}_{ij} = \left.\frac{\partial^2 T}{\partial \dot{q}_i \partial \dot{q}_j}\right|_{\vec{q}=0} \qquad \mathbf{K}_{ij} = \left.\frac{\partial^2 U}{\partial q_i \partial q_j}\right|_{\vec{q}=0}$$

Transverse oscillations of taut string: $k_T = \left.\frac{\text{tension}}{\text{length}}\right|_{\text{equilib}}$

Inner Product Space description and **Normal Coordinates** $\qquad$ * $\vec{x}$ = column vector of generalized coord

- **Space** : $|\vec{x}(t)\rangle \equiv$ solutions of a linear oscillator system
- **Inner Product** : $\langle \vec{y}|\vec{x}\rangle \equiv \vec{y}^T \mathbf{M}\vec{x}$ $\qquad$ and associated **magnitude** : $|\vec{x}|^2 \equiv \langle \vec{x}|\vec{x}\rangle$
- **Basis** : $|\hat{a}_m\rangle$ of eigenvectors defined by $\mathbf{K}\vec{a}_m = \omega_m^2 \mathbf{M}\vec{a}_m$ and normalization $\hat{a}_m \equiv \vec{a}_m / |\vec{a}_m|$
- **Basis is Orthonormal** : $\langle \hat{a}_n|\hat{a}_m\rangle = \delta_{nm}$
- **Completeness** for $\vec{x}(t)$ $\quad$ and $\quad$ **Normal Coordinates** ξ_m :

  ξ_m is the **component** of $\vec{x}$ along mode m : $\quad |\vec{x}(t)\rangle = \sum_{\text{modes } m} |\hat{a}_m\rangle\langle\hat{a}_m|\vec{x}(t)\rangle \equiv \sum_{\text{modes } m} \hat{a}_m \xi_m(t)$

  ξ_m is **projected out** of $\vec{x}$ by : $\quad \xi_m(t) = \langle \hat{a}_m|\vec{x}(t)\rangle = A_m \cos(\omega_m t - \delta_m)$
- **Transformation** betw x-space and ξ-space :

  vectors : $\quad \vec{\xi} = \mathbf{R}\vec{x} \qquad \vec{x} = \mathbf{R}^{-1}\vec{\xi} \qquad \mathbf{R}^{-1} = \begin{pmatrix} | & | & \\ \hat{a}_1 & \hat{a}_2 & \ldots \\ | & | & \end{pmatrix} \qquad \mathbf{R} = \left(\mathbf{R}^{-1}\right)^T \mathbf{M}$

  matrices : $\quad \mathbf{M}^{\xi} = \left(\mathbf{R}^{-1}\right)^T \mathbf{M}\mathbf{R}^{-1} \quad \rightarrow \quad \mathbf{M}^{\xi}_{mn} = \delta_{mn} \quad \& \quad \mathbf{K}^{\xi}_{mn} = \omega_m^2 \delta_{mn}$

  inhomogeneous EOM : $\quad \mathbf{M}\ddot{\vec{x}} + \mathbf{K}\vec{x} = \vec{F}$ in x-space $\quad \rightarrow \quad \mathbf{M}^{\xi}\ddot{\vec{\xi}} + \mathbf{K}^{\xi}\vec{\xi} = \left(\mathbf{R}^{-1}\right)^T \vec{F}$ in ξ-space

---

Fourier Series as an Inner Product Space

- **Space** : $|f\rangle \equiv$ **τ-periodic functions** $f(t)$ that are periodic over $t = [-\tau/2 \rightarrow \tau/2]$, with $\omega = 2\pi/\tau$
- **Basis #1** : $|n\rangle \equiv \begin{cases} \sin(n\omega t) & n = 1,\ldots,\infty \\ 1/\sqrt{2} & n = 0 \\ \cos(n\omega t) & n = -1,\ldots,-\infty \end{cases}$ $\qquad$ • **Inner Product #1** : $\langle g|f\rangle \equiv \frac{2}{\tau}\int_{-\tau/2}^{\tau/2} g(t)f(t)\,dt$
- **Basis #2** : $|n\rangle = e^{in\omega t}$ $\qquad$ • **Inner Product #2** : $\langle \tilde{g}|\tilde{f}\rangle \equiv \frac{1}{\tau}\int_{-\tau/2}^{\tau/2} g^*(t)f(t)\,dt$

➔ **Basis is Orthonormal** : $\langle n|m\rangle = \delta_{nm}$ $\qquad$ ➔ **Completeness** : any $|f\rangle = \sum_{n=-\infty}^{+\infty} |n\rangle\langle n|f\rangle$

Miscellanea

$$d\vec{l} = ds\,\hat{s} + s\,d\phi\,\hat{\phi} + dz\,\hat{z} \qquad d\vec{l} = dr\,\hat{r} + r\,d\theta\,\hat{\theta} + r\sin\theta\,d\phi\,\hat{\phi}$$

$$\vec{a} = \hat{s}\left[\ddot{s} - s\dot{\phi}^2\right] + \hat{\phi}\left[s\ddot{\phi} + 2\dot{s}\dot{\phi}\right] + \hat{z}\left[\ddot{z}\right]$$

$$\vec{a} = \hat{r}\left[\ddot{r} - r\dot{\theta}^2 - r\dot{\phi}^2\sin^2\theta\right] + \hat{\theta}\left[r\ddot{\theta} + 2\dot{r}\dot{\theta} - r\dot{\phi}^2\sin\theta\cos\theta\right] + \hat{\phi}\left[\sin\theta\left(r\ddot{\phi} + 2\dot{r}\dot{\phi}\right) + \cos\theta\left(2r\dot{\theta}\dot{\phi}\right)\right]$$

$$L = T - U \qquad \frac{\partial L}{\partial q_i} = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) \qquad H = \frac{\partial L}{\partial \dot{q}_i}\dot{q}_i - L$$

canonical/generalized : $p_i \equiv \frac{\partial L}{\partial \dot{q}_i} \qquad Q_i \equiv \frac{\partial L}{\partial q_i}$

2-Body Central Force Problems & Scattering

- **Coordinates & Reduced Mass** : $\vec{r}_1 = \vec{R} + \frac{m_2}{M}\vec{r}\,, \quad \vec{r}_2 = \vec{R} - \frac{m_1}{M}\vec{r}\,, \quad \vec{r} \equiv \vec{r}_1 - \vec{r}_2\,, \quad \mu = \frac{m_1 m_2}{M}$
- **Centrifugal** force & PE : $\vec{F}_{cf} = \frac{L^2}{\mu r^3}\hat{r}\,, \quad U_{cf} = \frac{L^2}{2\mu r^2}\,,$ effective radial $U^* = U + U_{\text{cf}}$
- **Angular EOM** : $\dot{\phi} = \frac{L}{\mu r^2}$ • **Radial EOMs** : $\mu\ddot{r} = F(r) + F_{\text{cf}}(r)\,, \quad E = T + U = \frac{1}{2}\mu\dot{r}^2 + U_{\text{cf}} + U$
- **Path Equation** : $u(\phi) \equiv 1/r(\phi) \;\rightarrow\; u'' + u = -\frac{\mu F(1/u)}{L^2 u^2}$ and $u' = -\frac{\mu\dot{r}}{L}$
- **Conics** : With (r, ϕ) centered on a focal point and E ≡ Ellipse, H ≡ Hyperbola

$$\frac{1}{r} = \frac{a}{b^2}(\pm 1 + e\cos\phi) \text{ with } \begin{cases} +\text{: E or H-near-branch} \\ -\text{: H-far-branch} \end{cases}, \qquad e = \frac{c}{a} = \frac{\sqrt{a^2 \mp b^2}}{a} \text{ with } \begin{cases} -\text{: E} \\ +\text{: H} \end{cases}$$

- **Kepler Orbits** $F = -\frac{\gamma}{r^2}$: $r(\phi) = \frac{r_0}{\text{sgn}[\gamma] + e\cos\phi}$ with $r_0 = \frac{L^2}{\mu|\gamma|} = \frac{b^2}{a} = a\left|1 - e^2\right|\,, \quad E = \mp\frac{|\gamma|}{2a} = \frac{|\gamma|(e^2 - 1)}{2r_0}$

  Bounded orbits: $\tau^2 = \frac{4\pi^2\mu}{\gamma}a^3\,, \quad r_0 = \frac{2\,r_{\text{min}}r_{\text{max}}}{r_{\text{min}} + r_{\text{max}}}\,, \quad e = \frac{r_{\text{max}} - r_{\text{min}}}{r_{\text{max}} + r_{\text{min}}}$

  Unbounded orbits: scattering angle $\theta = \pi - 2\alpha$ with $\tan\alpha = \frac{b}{a}\,,$ impact parameter $b =$ semi-minor axis b ☺

- **XSec** : $d\Omega \equiv \frac{dA}{r^2} = \begin{cases} \sin\theta\, d\theta\, d\phi \\ d\theta_x\, d\theta_y \end{cases}, \quad \frac{d\sigma}{d\Omega} = \frac{b}{\sin\theta}\left|\frac{db}{d\theta}\right|$ with $\theta =$ scattering angle • **Lumi**: $\mathcal{L} = n_A N_e\,, \quad \frac{dN_{\text{ev}}}{dt} = \mathcal{L}\sigma$

- **Earth Specifications**: $R_\oplus = 6.4 \times 10^6$ m, $GM_\oplus = gR_\oplus^2$, $g = 10$ m/s^2
- **Earth Orbit**: $r_0 = a = 1$ A.U. $= 1.5 \times 10^{11}$ m, orbital velocity $v_\oplus = 3 \times 10^4$ m/s
- **Atomic Data**: 1 amu ≈ mass of 1 nucleon (proton or neutron) = 1.66×10^{-27} kg = $(5/3) \times 10^{-24}$ g
  Gas at STP has $N_{\text{Avog}} = 6.02 \times 10^{23}$ molecules in 22.4 liters; 1 **barn** = 10^{-28} m^2 = 10^{-24} cm^2

General Relativity

$$d\tau^2 = dt^2(1 - 2M/r) - \frac{dr^2}{(1 - 2M/r)} - r^2 d\phi^2 \quad \text{with} \quad \begin{matrix} M \equiv GM_{kg}/c^2 \\ t \equiv t_{\text{sec}}c \end{matrix} \qquad d\sigma^2 = -d\tau^2 \qquad L_{SI} = -mc^2\frac{d\tau}{dt}$$

Inertia Tensor

$$I_{ij} = \int dm\,(\delta_{ij}r^2 - r_i r_j) \qquad \mathbf{I} = \int dm \begin{pmatrix} y^2+z^2 & -xy & -xz \\ \cdot & z^2+x^2 & -yz \\ \cdot & \cdot & x^2+y^2 \end{pmatrix}$$

Principal Axes : $\mathbf{I}\hat{e} = \lambda\hat{e}$ $\qquad \mathbf{I} = \mathbf{I}_{CM} + \mathbf{I}'$

$\vec{L}^{(B)} = \mathbf{I}^{(B)}\vec{\omega}$ ∀ body-fixed point B $\qquad T = \frac{1}{2}\vec{\omega}^T\mathbf{I}\vec{\omega}$

For $\vec{B}$ fixed in body frame, $\left.\frac{d\vec{B}}{dt}\right|_{\text{due to body rotation}} = \vec{\omega} \times \vec{B}$

$$\vec{\tau} = \dot{\vec{L}} = \left.\dot{\vec{L}}\right|_{\text{within body}} + \vec{\omega} \times \vec{L}$$

$$\tau_1 = I_1\dot{\omega}_1 + (I_3 - I_2)\omega_2\omega_3$$
$$\tau_2 = I_2\dot{\omega}_2 + (I_1 - I_3)\omega_3\omega_1$$
$$\tau_3 = I_3\dot{\omega}_3 + (I_2 - I_1)\omega_1\omega_2$$

* **I** simplifies for lamina, reflection symmetry, n-fold axisymmetry

Rotations: orthogonal $\mathbf{R}^T = \mathbf{R}^{-1}$

passive rotation from S frame $\{\hat{x}, \hat{y}, \hat{z}\}$ to S^* frame $\{\hat{e}_1, \hat{e}_2, \hat{e}_3\}$:
$$\mathbf{R}^{-1} = \begin{pmatrix} e_{1x} & e_{2x} & e_{3x} \\ e_{1y} & e_{2y} & e_{3y} \\ e_{1z} & e_{2z} & e_{3z} \end{pmatrix}$$

→ transforms: $\vec{v}^* = \mathbf{R}\vec{v}\,, \quad \mathbf{I}^* = \mathbf{R}\,\mathbf{I}\,\mathbf{R}^{-1}$

Free symmetric top : precession of $\vec{\omega}$ is
$\vec{\Omega}^* = \left[(I_3/I_1) - 1\right]\omega_3\hat{e}_3$ body, $\quad \vec{\Omega} = \vec{L}/I_1$ lab;
$\vec{L}, \vec{\omega}, \hat{e}_3$ always coplanar, $\quad \vec{\omega} = -\vec{\Omega}^* + \vec{\Omega}$