Damped / Driven Linear Oscillators from 325

EOM : $\ddot{x} + 2\beta\dot{x} + \omega_0^2 x = f(t)$ where $\vec{F}_{\text{restore}} = -k\,\vec{x} \equiv -m\,\omega_0{}^2\,\vec{x}$, $\vec{F}_{\text{damp}} = -m(2\beta)\vec{v}$, $\vec{F}_{\text{drive}} = m\,f(t)$

Damping Cases : Critical damping : $\beta = \omega_0 \equiv \sqrt{\dfrac{k}{m}}$, Underdamping: $\beta < \omega_0$, Overdamping: $\beta > \omega_0$

Coupled Linear Oscillators

$$\mathbf{M}\ddot{\vec{q}} = -\mathbf{K}\vec{q} \qquad T = \frac{1}{2}\mathbf{M}_{ij}\dot{q}_i\dot{q}_j \qquad U = \frac{1}{2}\mathbf{K}_{ij}q_iq_j \qquad \mathbf{M}_{ij} = \left.\frac{\partial^2 T}{\partial\dot{q}_i\partial\dot{q}_j}\right|_{\vec{q}=0} \qquad \mathbf{K}_{ij} = \left.\frac{\partial^2 U}{\partial q_i\partial q_j}\right|_{\vec{q}=0}$$

Transverse oscillations of taut string: $k_T = \left.\dfrac{\text{tension}}{\text{length}}\right|_{\text{equilib}}$

Inner Product Space description and **Normal Coordinates** * $\vec{x}$ = column vector of generalized coord

- **Space** : $|\vec{x}(t)\rangle \equiv$ solutions of a linear oscillator system
- **Inner Product** : $\langle\vec{y}|\vec{x}\rangle \equiv \vec{y}^T\mathbf{M}\,\vec{x}$ and associated **magnitude** : $|\vec{x}|^2 \equiv \langle\vec{x}|\vec{x}\rangle$
- **Basis** : $|\hat{a}_m\rangle$ of eigenvectors defined by $\mathbf{K}\,\vec{a}_m = \omega_m^2\mathbf{M}\,\vec{a}_m$ and normalization $\hat{a}_m \equiv \vec{a}_m / |\,\vec{a}_m\,|$
- **Basis is Orthonormal** : $\langle\hat{a}_n|\hat{a}_m\rangle = \delta_{nm}$
- **Completeness** for $\vec{x}(t)$ and **Normal Coordinates** ξ_m :

  ξ_m is the **component** of $\vec{x}$ along mode m : $|\vec{x}(t)\rangle = \sum_{\text{modes } m} |\hat{a}_m\rangle\langle\hat{a}_m|\vec{x}(t)\rangle \equiv \sum_{\text{modes } m} \hat{a}_m\,\xi_m(t)$

  ξ_m is **projected out** of $\vec{x}$ by : $\xi_m(t) = \langle\hat{a}_m|\vec{x}(t)\rangle = A_m\cos(\omega_m t - \delta_m)$
- **Transformation** betw x-space and ξ-space :

  vectors : $\vec{\xi} = \mathbf{R}\,\vec{x}$ $\quad \vec{x} = \mathbf{R}^{-1}\,\vec{\xi}$ $\quad \mathbf{R}^{-1} = \begin{pmatrix} | & | & \\ \hat{a}_1 & \hat{a}_2 & \ldots \\ | & | & \end{pmatrix}$ $\quad \mathbf{R} = \left(\mathbf{R}^{-1}\right)^T\mathbf{M}$

  matrices : $\mathbf{M}^{\xi} = \left(\mathbf{R}^{-1}\right)^T\mathbf{M}\,\mathbf{R}^{-1}$ $\rightarrow$ $\mathbf{M}^{\xi}_{mn} = \delta_{mn}$ & $\mathbf{K}^{\xi}_{mn} = \omega_m^2\delta_{mn}$

Lagrangian Mech from 325 * Gen. coord q_i must be indep

$S \equiv \int_{t_1,\vec{q}_1}^{t_1,\vec{q}_1} dt\,L(q_i,\dot{q}_i,t)$ $\qquad \delta S = 0 \rightarrow \dfrac{\partial L}{\partial q_i} = \dfrac{d}{dt}\left(\dfrac{\partial L}{\partial\dot{q}_i}\right)$ for each q_i

$H \equiv \dot{q}_i\,(\partial L/\partial\dot{q}_i) - L$ conserved when $\partial L/dt = 0$

Principle of Least Action :

$L = T - U \rightarrow \delta S = 0$ @ true $\{q_i(t)\}$

Gen. force $Q_i \equiv \dfrac{\partial L}{\partial q_i}$, momentum $p_i \equiv \dfrac{\partial L}{\partial\dot{q}_i}$

$H \equiv p_i\dot{q}_i - L$ equals T+U when $\vec{r}_a = \vec{r}_a(q_i)$