2-Body Central Force Problems & Scattering

- **Coordinates & Reduced Mass** : $\vec{r}_1 = \vec{R} + \frac{m_2}{M}\vec{r}$, $\vec{r}_2 = \vec{R} - \frac{m_1}{M}\vec{r}$, $\vec{r} \equiv \vec{r}_1 - \vec{r}_2$, $\mu = \frac{m_1 m_2}{M}$

- **Centrifugal** force & PE : $\vec{F}_{cf} = \frac{L^2}{\mu r^3}\hat{r}$, $U_{cf} = \frac{L^2}{2\mu r^2}$, effective radial $U^* = U + U_{\text{cf}}$

- **Angular EOM** : $\dot{\phi} = \frac{L}{\mu r^2}$ • **Radial EOMs** : $\mu\ddot{r} = F(r) + F_{\text{cf}}(r)$, $E = T + U = \frac{1}{2}\mu\dot{r}^2 + U_{\text{cf}} + U$

- **Path Equation** : $u(\phi) \equiv 1/r(\phi) \;\rightarrow\; u'' + u = -\frac{\mu F(1/u)}{L^2 u^2}$ and $u' = -\frac{\mu\dot{r}}{L}$

- **Conics** : With (r, ϕ) centered on a focal point and E ≡ Ellipse, H ≡ Hyperbola

$$\frac{1}{r} = \frac{a}{b^2}(\pm 1 + e\cos\phi) \text{ with } \begin{cases} +\text{: E or H-near-branch} \\ -\text{: H-far-branch} \end{cases}, \qquad e = \frac{c}{a} = \frac{\sqrt{a^2 \mp b^2}}{a} \text{ with } \begin{cases} -\text{: E} \\ +\text{: H} \end{cases}$$

- **Kepler Orbits** $F = -\frac{\gamma}{r^2}$: $r(\phi) = \frac{r_0}{\text{sgn}[\gamma] + e\cos\phi}$ with $r_0 = \frac{L^2}{\mu|\gamma|} = \frac{b^2}{a} = a\left|1 - e^2\right|$, $E = \mp\frac{|\gamma|}{2a} = \frac{|\gamma|(e^2 - 1)}{2r_0}$

  Bounded orbits: $\tau^2 = \frac{4\pi^2\mu}{\gamma}a^3$, $r_0 = \frac{2\, r_{\min} r_{\max}}{r_{\min} + r_{\max}}$, $e = \frac{r_{\max} - r_{\min}}{r_{\max} + r_{\min}}$

  Unbounded orbits: scattering angle $\theta = \pi - 2\alpha$ with $\tan\alpha = \frac{b}{a}$, impact parameter b = semi-minor axis b ☺

- **XSec** : $d\Omega \equiv \frac{dA}{r^2} = \begin{cases} \sin\theta\, d\theta\, d\phi \\ d\theta_x\, d\theta_y \end{cases}$, $\frac{d\sigma}{d\Omega} = \frac{b}{\sin\theta}\left|\frac{db}{d\theta}\right|$ with θ = scattering angle • **Lumi**: $\mathcal{L} = n_A N_e$, $\frac{dN_{\text{ev}}}{dt} = \mathcal{L}\sigma$

- **Earth Specifications**: $R_\oplus = 6.4 \times 10^6$ m, $GM_\oplus = gR_\oplus^2$, $g = 10$ m/s²
- **Earth Orbit**: $r_0 = a = 1$ A.U. $= 1.5 \times 10^{11}$ m, orbital velocity $v_\oplus = 3 \times 10^4$ m/s
- **Atomic Data**: 1 amu ≈ mass of 1 nucleon (proton or neutron) = 1.66×10^{-27} kg = $(5/3) \times 10^{-24}$ g
  Gas at STP has $N_{\text{Avog}} = 6.02 \times 10^{23}$ molecules in 22.4 liters; 1 **barn** = 10^{-28} m² = 10^{-24} cm²

Inertia Tensor

$$I_{ij} = \int dm\; (\delta_{ij} r^2 - r_i r_j) \qquad \mathbf{I} = \int dm \begin{pmatrix} y^2+z^2 & -xy & -xz \\ \cdot & z^2+x^2 & -yz \\ \cdot & \cdot & x^2+y^2 \end{pmatrix}$$

Principal Axes : $\mathbf{I}\,\hat{e} = \lambda\,\hat{e}$ $\qquad \mathbf{I} = \mathbf{I}_{CM} + \mathbf{I}'$

$\vec{L}^{(B)} = \mathbf{I}^{(B)}\vec{\omega}$ ∀ body-fixed point B $\qquad T = \frac{1}{2}\vec{\omega}^T\mathbf{I}\vec{\omega}$

For $\vec{B}$ fixed in body frame, $\qquad \vec{\tau} = \dot{\vec{L}} = \dot{\vec{L}}\Big|_{\text{within body}} + \vec{\omega}\times\vec{L}$

$$\left.\frac{d\vec{B}}{dt}\right|_{\text{due to body rotation}} = \vec{\omega}\times\vec{B}$$

$$\tau_1 = I_1\dot{\omega}_1 + (I_3 - I_2)\omega_2\omega_3$$
$$\tau_2 = I_2\dot{\omega}_2 + (I_1 - I_3)\omega_3\omega_1$$
$$\tau_3 = I_3\dot{\omega}_3 + (I_2 - I_1)\omega_1\omega_2$$

* **I** simplifies for lamina, reflection symmetry, n-fold axisymmetry

Rotations: orthogonal $\mathbf{R}^T = \mathbf{R}^{-1}$

passive rotation from S frame $\{\hat{x}, \hat{y}, \hat{z}\}$ to S^*frame $\{\hat{e}_1, \hat{e}_2, \hat{e}_3\}$:

$$\mathbf{R}^{-1} = \begin{pmatrix} e_{1x} & e_{2x} & e_{3x} \\ e_{1y} & e_{2y} & e_{3y} \\ e_{1z} & e_{2z} & e_{3z} \end{pmatrix}$$

→ transforms: $\vec{v}^* = \mathbf{R}\,\vec{v}$, $\mathbf{I}^* = \mathbf{R}\,\mathbf{I}\,\mathbf{R}^{-1}$

Free symmetric top : precession of $\vec{\omega}$ is

$\vec{\Omega}^* = \left[\left(I_3/I_1\right) - 1\right]\omega_3\hat{e}_3$ body, $\vec{\Omega} = \vec{L}/I_1$ lab;

$\vec{L}, \vec{\omega}, \hat{e}_3$ always coplanar, $\qquad \vec{\omega} = -\vec{\Omega}^* + \vec{\Omega}$