

Derivation of $\vec{L}^{(B)} = \mathbf{I}^{(B)} \vec{\omega}$

$$\vec{L}^{(B)} = \int \vec{r}^{(B)} \times dm \dot{\vec{r}}^{(B)}$$

$\vec{r}^{(B)}$ points from body-fixed ref. point (B) to each piece of mass dm , so

$$\vec{r}^{(B)} \text{ is body-fixed } \rightarrow \dot{\vec{r}}^{(B)} = \vec{\omega} \times \vec{r}^{(B)}$$

$$\begin{aligned} \vec{L}^{(B)} &= \int dm \vec{r}^{(B)} \times (\vec{\omega} \times \vec{r}^{(B)}) \\ &= \int dm \vec{\omega} (\vec{r}^{(B)} \cdot \vec{r}^{(B)}) - \vec{r}^{(B)} (\vec{r}^{(B)} \cdot \vec{\omega}) \\ &= \int dm \vec{\omega} |r^{(B)}|^2 - \vec{r}^{(B)} \sum_j r_j^{(B)} \omega_j \end{aligned}$$

$$\begin{aligned} L_i^{(B)} &= \int dm \omega_i |r^{(B)}|^2 - r_i^{(B)} \sum_j r_j^{(B)} \omega_j \\ &= \sum_j \int dm \omega_j \delta_{ij} |r^{(B)}|^2 - r_i^{(B)} r_j^{(B)} \omega_j \\ &= \sum_j I_{ij}^{(B)} \omega_j \text{ with } \boxed{I_{ij}^{(B)} = \int dm \delta_{ij} |r^{(B)}|^2 - r_i^{(B)} r_j^{(B)}} \end{aligned}$$

$$\mathbf{I} = \int dm \begin{pmatrix} y^2+z^2 & -xy & -xz \\ \cdot & z^2+x^2 & -yz \\ \cdot & \cdot & x^2+y^2 \end{pmatrix} \text{ with implicit (B) ref. pt. on everything}$$

Derivation of Euler's Equations

$$\vec{\tau}^{(A)} = \frac{d\vec{L}^{(A)}}{dt} \text{ expressed in Body Frame } \mathbf{S}^* = \{\hat{e}_1, \hat{e}_2, \hat{e}_3\}$$

$$\vec{\tau} = \frac{d}{dt} (\mathbf{I} \vec{\omega}) = \frac{d}{dt} (I_1 \omega_1 \hat{e}_1 + I_2 \omega_2 \hat{e}_2 + I_3 \omega_3 \hat{e}_3)$$

Two sources of time-dep: (1) ω_i and (2) \hat{e}_i

$$\begin{aligned} \vec{\tau} &= I_1 \dot{\omega}_1 \hat{e}_1 + I_2 \dot{\omega}_2 \hat{e}_2 + I_3 \dot{\omega}_3 \hat{e}_3 \\ &\quad + I_1 \omega_1 (\vec{\omega} \times \hat{e}_1) + I_2 \omega_2 (\vec{\omega} \times \hat{e}_2) + I_3 \omega_3 (\vec{\omega} \times \hat{e}_3) \end{aligned}$$

$$\vec{\omega} \times \hat{e}_1 = (\omega_1 \hat{e}_1 + \omega_2 \hat{e}_2 + \omega_3 \hat{e}_3) \times \hat{e}_1 = 0 - \omega_2 \hat{e}_3 + \omega_3 \hat{e}_2$$

$$\vec{\omega} \times \hat{e}_2 = (\omega_1 \hat{e}_1 + \omega_2 \hat{e}_2 + \omega_3 \hat{e}_3) \times \hat{e}_2 = +\omega_1 \hat{e}_3 + 0 - \omega_3 \hat{e}_1$$

$$\vec{\omega} \times \hat{e}_3 = (\omega_1 \hat{e}_1 + \omega_2 \hat{e}_2 + \omega_3 \hat{e}_3) \times \hat{e}_3 = -\omega_1 \hat{e}_3 + \omega_2 \hat{e}_1 + 0$$

$$\tau_1 = I_1 \dot{\omega}_1 + \omega_2 \omega_3 (I_3 - I_2)$$

$$\tau_2 = I_2 \dot{\omega}_2 + \omega_3 \omega_1 (I_1 - I_3)$$

$$\tau_3 = I_3 \dot{\omega}_3 + \omega_1 \omega_2 (I_2 - I_1)$$

cycle
1 → 2 → 3