

## Phys 326 Discussion 2 – Weakly Coupled Oscillators, Part II

Last week, you calculated the general solution for the coupled-oscillator demo from class. The system consisted of a support-spring  $K$  whose top end was attached to a fixed point and whose bottom end was attached to a massless horizontal cross-bar. Two springs  $k$  (with  $k \ll K$ ) were suspended from either side of this cross-bar and a mass  $m$  was placed at the bottom end of each spring. We defined the  $x$  direction pointing downwards, and defined coordinates  $x_1$  and  $x_2$  for the deviation of each mass from its equilibrium position.

After defining the useful symbols  $\omega_0^2 \equiv \frac{k}{m}$  and  $\eta \equiv \frac{k}{K} \ll 1$ , you found the system's two eigenfrequencies:

$$\omega_s \approx \omega_0(1 - \eta) \text{ and } \omega_F = \omega_0 \quad \text{using the matrices } \mathbf{M} = \begin{pmatrix} m & 0 \\ 0 & m \end{pmatrix} \text{ and } \mathbf{K} \approx k \begin{pmatrix} 1 - \eta & -\eta \\ -\eta & 1 - \eta \end{pmatrix}$$

The system's general solution was:

$$\vec{x}(t) \equiv \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} A_S \cos(\omega_s t - \delta_S) + \begin{pmatrix} 1 \\ -1 \end{pmatrix} A_F \cos(\omega_F t - \delta_F).$$

### Problem 1 : The Distinctive Behavior of Weakly Coupled Oscillators

*Checkpoints<sup>1</sup>*

The normal modes of our system have an unusual feature: since  $\eta \ll 1$ , the normal frequencies  $\omega_F = \omega_0$  and  $\omega_s \approx \omega_0(1 - \eta)$  are almost the same. This is a characteristic feature of **weakly coupled oscillators** and it leads to a distinctive behavior. Note: you can find another example in Taylor section 11.3.

(a) When working with near-equal frequencies, there is a very useful “trick” that greatly simplifies things: introduce new frequency parameters that “split the difference” between the two modes. Here's what I mean:

- define the mean frequency  $\bar{\omega} \equiv (\omega_s + \omega_F) / 2$  and
- define the half-difference frequency  $\varepsilon$  so that  $\omega_s = \bar{\omega} - \varepsilon$  and  $\omega_F = \bar{\omega} + \varepsilon$

Write down expressions for  $\bar{\omega}$  and  $\varepsilon$  in terms of  $\omega_0$  and  $\eta$ .

(b) Start the system at time  $t = 0$  with both masses at rest at these positions:  $(x_1, x_2) = (b, 0)$ . Obtain the specific solution for  $x_1(t)$  and  $x_2(t)$  given these initial conditions. Remember to use the new frequency parameters  $\bar{\omega}$  and  $\varepsilon$  instead of  $\omega_s$  and  $\omega_F$ .

(c) Here are two extremely useful trig relations:

$$\frac{1}{2} [\cos(a - b) + \cos(a + b)] = \cos a \cos b \quad \text{and} \quad \frac{1}{2} [\cos(a - b) - \cos(a + b)] = \sin a \sin b$$

(They can be trivially derived from the cosine-addition formula, or using complex numbers.) Apply these relations to rewrite  $x_1(t)$  and  $x_2(t)$  as products of two trig functions each.

<sup>1</sup> (a)  $\bar{\omega} = \omega_0(1 - \frac{1}{2}\eta)$  and  $\varepsilon = \omega_0\eta / 2$  (b)  $x_1(t) = \frac{1}{2}b [\cos(\bar{\omega}t - \varepsilon t) + \cos(\bar{\omega}t + \varepsilon t)]$  &  $x_2(t) = \frac{1}{2}b [\cos(\bar{\omega}t - \varepsilon t) - \cos(\bar{\omega}t + \varepsilon t)]$   
 (c)  $x_1(t) = b \cos(\bar{\omega}t) \cos(\varepsilon t)$  &  $x_2(t) = b \sin(\bar{\omega}t) \sin(\varepsilon t)$  (e)  $\eta = 1/40$  Note: only-1  $\rightarrow$  only-2  $\rightarrow$  only-1 is a *half cycle* of  $\sin(\varepsilon t)$ .  
 (f)  $\xi_+(t)$  oscillates at the slow frequency only,  $\xi_-(t)$  at the fast frequency only (g)  $\xi_+$  = CM position;  $\xi_-$  = position of mass 1 relative to the CM (mass 2 is the same distance away from the CM, just on the other side) (h)  $\tau = 391$   $\tau_S = 401$   $\tau_F \approx 400$  sec  
 (i)  $x_1^p = x_2^p = gm / [k(1 - 2\eta)] = g / \omega_s^2$ , i.e. a constant! Gravity simply applies a constant downward shift to  $x_1(t)$  and  $x_2(t)$ .

(d) Now the interesting part: what are these solution forms telling us?? To figure it out, plot  $x_1$  and  $x_2$  versus time. To figure out how these graphs should look, two suggestions: [1] Remember that the frequency  $\varepsilon$  is very small compared to  $\bar{\omega}$ , so  $\cos(\varepsilon t)$  has a very long period while  $\cos(\bar{\omega} t)$  has a very short one by comparison. [2] Think of the word envelope  $\rightarrow$  when a sin or cos is multiplied with another function  $f$ , it imposes an *envelope* on  $f$ :  $f$  is left alone when the trig function is 1, inverted when the trig function is  $-1$ , and sinusoidally *squashed* at all points in between. Plot the quickly-varying function first, then mentally multiply it by the slowly-varying function to produce the final result.

(e) We did this very experiment in class: we started mass 1 oscillating while leaving mass 2 at rest. What we found was that the oscillation *moved over* to mass 2 after a while, then returned to mass 1. In class we timed this only-1-oscillating  $\rightarrow$  only-2-oscillating  $\rightarrow$  only-1-oscillating sequence and found that it took approximately 40 seconds. In contrast, when only one mass was oscillating (only one spring active), it completed about one cycle per second. Given this information, calculate the quantity  $\eta = k / K$  for our demo.

☞ The phenomenon you've just analyzed is the familiar slow **beating** that occurs when two waves of very similar frequencies are superposed: they add constructively, then destructively, then constructively again, with this interference varying on a much slower timescale than the waves themselves.

(f) Our demo provides a good opportunity to gain some intuition about **normal coordinates**. We know from our work last week that the normal coordinates for this system are  $\xi_+ = x_1 + x_2$  and  $\xi_- = x_1 - x_2$ , i.e. these are the coordinates that decouple the equations of motion. Find  $\xi_+(t)$  and  $\xi_-(t)$  for the initial condition that, at  $t=0$ , the system starts from rest at position  $(x_1, x_2) = (b, 0)$ . This is the same initial condition as in part (b), so you can use your earlier work to answer this.

(g) Make a very rough plot of  $\xi_+(t)$  and  $\xi_-(t)$ , just enough to compare with your  $x_{1,2}(t)$  plots from part (d). See how differently – and nicely! – the normal coordinates behave? ☺ Now let's interpret them! For this particular system, see if you can come up with a physical interpretation in words for  $\xi_+$  and for  $\xi_-$ .

Hint: the phrase “center of mass” may be helpful. Note: The normal coordinates are not always as readily interpretable as they are for this system, but usually *some* of them are.

(h) Is our system **periodic**? Suppose the exact value of  $\eta$  was  $1 / 40.1$ . By considering the normal frequencies  $\omega_0$  and  $\omega_0(1 - \eta)$ , where  $\omega_0 = 2\pi / (1\text{sec})$ , figure out the overall period of the system. By overall period, we mean the time  $\tau$  it takes for the system to go from a certain state = particular values of the positions  $\vec{x}$  and velocities  $\dot{\vec{x}}$  back to the exact same state. Hint:  $\eta = 1/40.1$  can be written  $10/401$ .

Woah! We saw with our own eyes the system go through the cycle only-1-oscillating  $\rightarrow$  only-2-oscillating  $\rightarrow$  only-1-oscillating every 40 seconds or so ... but as you see from your calculation, the actual period of the system could be *much* longer than that. To understand why, look at your part (d) plots: mass 2's oscillation amplitude goes to zero every time  $\varepsilon t = n\pi$ , which is every 40 seconds, but mass 1 may not be doing the exact same thing at  $t = 0$  and  $t = \pi/\varepsilon = 40$  sec. It's possible that the entire system isn't even periodic at all.

(i) Finally let's **add GRAVITY**. This means adding a driving force to our coupled system ... but it is a very simple force: a constant! Our equations of motion become  $\mathbf{M}\ddot{\vec{x}} + \mathbf{K}\vec{x} = mg \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ . This is an inhomogeneous

equation, but we know how to deal with that: find a particular solution  $\vec{x}^P(t)$  that satisfies it, add in the general homogeneous solution we've already computed, and we're done! What particular solution would work here? Think SIMPLE. A REALLY simple  $\vec{x}^P$  will work. ☺ Does uniform gravity affect the normal modes?