Phys 326 Discussion 4 – Normal Coordinates, Inner Product Spaces, and Degeneracy

Here is a summary of our description of normal mode solutions as an inner product space:

- **Space** : $|\vec{x}(t)\rangle \equiv$ solutions of a linear oscillator system in terms of generalized coordinates $\vec{x}$
- **Inner Product** : $\boxed{\langle \vec{y}|\vec{x}\rangle \equiv \vec{y}^T \mathbf{M}\,\vec{x}}$ and associated **magnitude** : $|\vec{x}|^2 \equiv \langle \vec{x}|\vec{x}\rangle$
- **Basis** : $|\hat{a}_m\rangle$ of eigenvectors defined by $\mathbf{K}\vec{a}_m = \omega_m^2 \mathbf{M}\vec{a}_m$ and normalized by $\hat{a}_m = \vec{a}_m / |\vec{a}_m|$
- **Basis is Orthonormal** : $\langle \hat{a}_n|\hat{a}_m\rangle = \delta_{nm}$
- **Completeness** for $\vec{x}(t)$ and associated definition of **Normal Coordinates** ξ_m :

  ξ_m is the **component** of $\vec{x}$ along mode m : $|\vec{x}(t)\rangle = \sum_{\text{modes } m} |\hat{a}_m\rangle\langle \hat{a}_m|\vec{x}(t)\rangle \equiv \sum_{\text{modes } m} \hat{a}_m\,\xi_m(t)$

  $\therefore$ definition of normal coordinates is : $\boxed{\xi_m(t) = \langle \hat{a}_m|\vec{x}(t)\rangle} = A_m \cos(\omega_m t - \delta_m)$

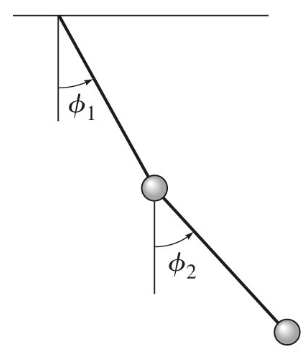
Apart from the elegance of this formalism, **normal coordinates** can be a useful solving technique because they **decouple** the problem by modes.

(1) The equations of motion are $\mathbf{M}_{ki}\ddot{x}_i = -\mathbf{K}_{kj}x_j$ in *x*-space, with each of the ODEs involving in general *all* of the coordinates x_i. In ξ-space, the EOMs decouple to $\ddot{\xi}_m = -\omega_m^2 \xi_m$: one separated ODE for each normal coordinate ξ_m. If our system has drag forces or driving forces to complicate the EOMs, we *must* decouple them, or we will not be able to apply our damped/driven oscillator solution techniques from PHYS 325 / MATH 285. Switching to normal coordinates is typically essential in these cases.

(2) On a less essential note, initial conditions are usually easier to deal with in ξ-space. Why? The normal coordinates decouple not only the EOMs but also their *solutions* by modes: each normal-coordinate solution is $\xi_m(t) = A_m \cos(\omega_m t - \delta_m) = \tilde{A}_m e^{i\omega_m t}$ or equivalently $B_m \cos(\omega_m t) + C_m \sin(\omega_m t)$, so it has 2 adjustable parameters that are completely independent (!!) of all the other adjustable parameters in your *n*-dimensional system.

Problem 1 : Normalized Basis & Normal Coordinates for Double Pendulum *Checkpoints*[1]

Let's explore our new concepts using the double pendulum, where the {upper, lower} pendula have lengths $\{l_1, l_2\}$, attached masses $\{m_1, m_2\}$, and make angles $\{\phi_1, \phi_2\}$ with the vertical. Using ϕ_1, ϕ_2 as our generalized coordinates, the mass and spring matrices for small oscillations of the general double pendulum are:



$$\mathbf{M} = m_1 l_1^2 \begin{pmatrix} 1+\alpha & \alpha\lambda \\ \alpha\lambda & \alpha\lambda^2 \end{pmatrix} \quad \& \quad \mathbf{K} = m_1 l_1 g \begin{pmatrix} 1+\alpha\lambda & 0 \\ 0 & \alpha\lambda \end{pmatrix} \quad \text{where } \alpha \equiv \frac{m_2}{m_1} \ \& \ \lambda \equiv \frac{l_2}{l_1}$$

(a) Using all the tricks in your bag, find the normal modes (frequencies and eigenvectors) for this particular double-pendulum configuration:

---

[1] (a) $\omega_S^2 = 4g/3$, $\omega_F^2 = 4g$, $\vec{a}_S = \begin{pmatrix}1\\2\end{pmatrix}$, $\vec{a}_F = \begin{pmatrix}1\\-2\end{pmatrix}$ (b) $\langle \vec{a}_S|\vec{a}_F\rangle = \vec{a}_S^T\,\mathbf{M}\,\vec{a}_F = 0$, yep! (c) $\hat{a}_S = \frac{1}{\sqrt{3}}\begin{pmatrix}1\\2\end{pmatrix}$, $\hat{a}_F = \begin{pmatrix}1\\-2\end{pmatrix}$

(d) $\xi_S(t) = \tilde{A}_S e^{i\omega_S t}$, $\xi_F(t) = \tilde{A}_F e^{i\omega_F t}$ (e) $\xi_S = (2\phi_1 + \phi_2)\sqrt{3}/4$, $\xi_F = (2\phi_1 - \phi_2)/4$ (f) Hint: the EOMs are the rows of $\mathbf{M}\ddot{\vec{\phi}} = -\mathbf{K}\vec{\phi}$; if that is not 100% clear to you, ask your instructor. Transformed EOMs: $\ddot{\xi}_S = -(4g/3)\xi_S$, $\ddot{\xi}_F = -4g\,\xi_F$

(h) Hint: first find the t=0 values for $\xi_{S,F}$ and $\dot{\xi}_{S,F}$ … you then get $B_S = 3\sqrt{3}/2$, $B_F = 1/2$, $C_S = 3\sqrt{3}/(8\sqrt{g})$, $C_F = -1/(8\sqrt{g})$.

Now you have the solution in ξ–space, $\xi_{S,F}(t) \rightarrow$ switch back to angles to obtain the final solution $\vec{\phi}(t) = \xi_S(t)\,\hat{a}_S + \xi_F(t)\,\hat{a}_F$.

$$m_1 = 3, \qquad m_2 = 1, \qquad l_1 = l_2 = \frac{1}{2} \quad \rightarrow \quad \mathbf{M} = \frac{1}{4}\begin{pmatrix} 4 & 1 \\ 1 & 1 \end{pmatrix} \quad \text{and} \quad \mathbf{K} = \frac{1}{4}\begin{pmatrix} 8g & 0 \\ 0 & 2g \end{pmatrix}$$

(b) Take the good old dot-product of the eigenvectors $\vec{a}_S$ (slow mode) and $\vec{a}_F$ (fast mode). You should find that it is not zero $\rightarrow$ $\vec{a}_S$ and $\vec{a}_F$ are not orthogonal using that definition of the inner product! Next, verify that $\vec{a}_S$ and $\vec{a}_F$ ARE orthogonal using the new inner product we derived (first box on page 1). FYI: If you were very astute, you may have already used the orthogonality relation to *find* one of the eigenvectors; if so, bravo!

(c) Use your new skills to normalize the eigenvectors, i.e. to obtain $\hat{a}_S$ and $\hat{a}_F$.

We now turn to the **normal coordinates** ξ_S and ξ_F for this system. Until now, we have only used normal coordinates as a trick for solving 2-DOF systems that are symmetric under the exchange of the two coordinates, by decoupling the equations of motion. Well, a complete set $\xi_{1,\ldots,}\xi_n$ can be obtained for *all* linear oscillator problems, and they *always* decouple the *n* equations of motion. You can regard that as their definition: the ξ's are the coordinates that yield *n* completely decoupled EOMs. Unfortunately, it is generally not possible to guess what they are in advance, so they are only useful as a trick for *finding* the normal modes in a few simple cases. But the normal coordinates have other useful properties, so let's explore them!

(d) As we know, the general solution for our double pendulum is the superposition of the two normal modes: $\vec{\phi}(t) = \tilde{A}_S e^{i\omega_S t}\,\hat{a}_S + \tilde{A}_F e^{i\omega_F t}\,\hat{a}_F$. Using the definition $\xi_m(t) = \langle \hat{a}_m | \vec{\phi}(t) \rangle$ (second box on page 1) and your normalized eigenvectors, determine the $\xi_S(t)$ and $\xi_F(t)$ as a function of time. Do you see how they are the *components* of $\vec{\phi}(t)$ in our $\hat{a}_n$ basis? Do you see how each $\xi_m(t)$ gives the behaviour of a *single mode m*?

(e) Now use the definition $\xi_m = \langle \hat{a}_m | \vec{\phi} \rangle$ in a different way: instead of dropping in the full time-dependent solution $\vec{\phi}(t)$ on the right-hand side of that inner product, just drop in the coordinate vector $\vec{\phi} = (\phi_1, \phi_2)$. This time you will obtain ξ_S and ξ_F as a function of your generalized coordinates ϕ_1 and ϕ_2.

(f) You just found the transformation from our angle coordinates ϕ_1, ϕ_2 to the normal coordinates ξ_S, ξ_F. Do these new coordinates really give decoupled EOMs, as advertised? Let's find out! Write down the two equations of motion in terms of angles, then add and subtract them judiciously to transform them to normal coordinates. What new EOMs do you get? Important: you can read off the EOMs immediately from **M** and **K**; if you're not sure how, see the checkpoint.

(g) We now have two coordinate systems, and so two ways of writing the general solution for our system:

$$\begin{pmatrix} \phi_1(t) \\ \phi_2(t) \end{pmatrix} = \tilde{A}_S\, e^{i\omega_S t}\begin{pmatrix} 1/\sqrt{3} \\ 2/\sqrt{3} \end{pmatrix} + \tilde{A}_F\, e^{i\omega_F t}\begin{pmatrix} 1 \\ -2 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} \xi_S(t) \\ \xi_F(t) \end{pmatrix} = \tilde{\alpha}_S\, e^{i\omega_S t}\begin{pmatrix} 1 \\ 0 \end{pmatrix} + \tilde{\alpha}_F\, e^{i\omega_F t}\begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

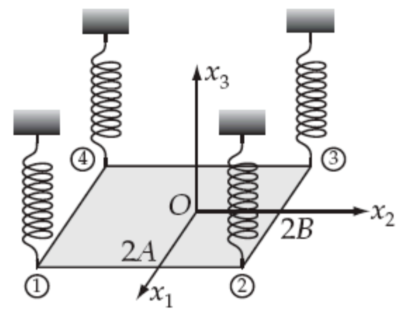
These are important expressions … to study them further, use your accumulated knowledge to demonstrate that the $\tilde{A}_{S,F}$ and $\tilde{\alpha}_{S,F}$ coefficients are EXACTLY THE SAME.

(h) Normal coordinates are the best way to deal with initial conditions. The general solution is:

$$\begin{pmatrix} \xi_S(t) \\ \xi_F(t) \end{pmatrix} = \begin{pmatrix} \tilde{A}_S\, e^{i\omega_S t} \\ \tilde{A}_F\, e^{i\omega_F t} \end{pmatrix} = \begin{pmatrix} A_S \cos(\omega_S t - \delta_S) \\ A_F \cos(\omega_F t - \delta_S) \end{pmatrix} = \begin{pmatrix} B_S \cos(\omega_S t) + C_S \sin(\omega_S t) \\ B_F \cos(\omega_F t) + C_F \sin(\omega_F t) \end{pmatrix}$$

That last form is *ideal* for initial conditions specified at $t = 0$. Use it and part (e) to fit the B's and C's that match these initial conditions: at $t = 0$, $\phi_1 = \phi_2 = 2$ while $\dot{\phi}_1 = 0$ and $\dot{\phi}_2 = 1$. That gives you $\xi_S(t)$ and $\xi_F(t)$ … how do you use those functions to build your final solutions $\phi_1(t)$ and $\phi_2(t)$?

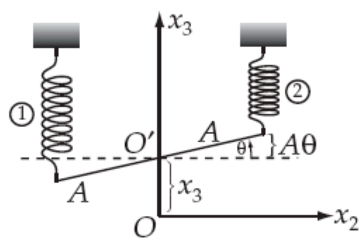
Problem 2 : The Degenerate Modes of a Suspended Plate *Checkpoints*[2]



A thin, flat, homogeneous plate has mass M and lies in the x_1-x_2 plane with its center at the origin. The plate's sides have length $2A$ in the x_2 direction and $2B$ in the x_1 direction. The plate is suspended from a fixed support by four springs of equal spring-constant k at the four corners of the plate. The top figure shows the equilibrium configuration of this system.

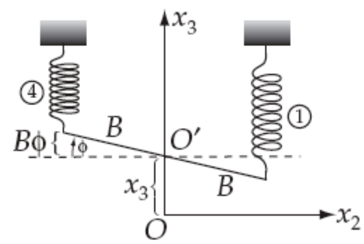
We have three degrees of freedom:

1. vertical motion, with the center of the plate moving along $\pm\hat{x}_3$
2. tipping motion around the $\hat{x}_1$ axis, described by the angle θ
3. tipping motion around the $\hat{x}_2$ axis, described by the angle ϕ



Our generalized coordinates for this problem are thus $\vec{q} = \{x_3, \theta, \phi\}$;
the angles remain small to ensure a linear system. The plate's moments of inertia are $I_1 = ⅓\, MA^2$ around $\hat{x}_1$ and $I_2 = ⅓\, MB^2$ around $\hat{x}_2$.

(a) The **M** and **K** matrices and resulting eigenfrequencies are given in the checkpoint. Come back later and derive them for practice, but for now just grab them. The interesting feature of this problem is that two of the modes are **degenerate**, meaning two of the eigenfrequencies are *the same*.



(b) To find out how this plate likes to move, you must find the eigenvector $\vec{a}$ for each mode. Because of the degeneracy, you must make some arbitrary choice when building $\vec{a}$ for the two modes with the same frequency. Use this common tactic: set one of the free components of $\vec{a}$ to zero for one of the degenerate modes, then figure out the other $\vec{a}$ using orthogonality. Finally, normalize your eigenvectors.

(c) Find the system's normal coordinates and write down their solutions for the case when the system starts at rest from positions $\vec{q}(t=0) = \{x_{30}, \theta_0, \phi_0\}$. Note: you will see that in the case of this plate, normal coordinates are no easier to work with than the regular ones. But practice is good. ☺

(d) Finally, add to the plate a thin bar of mass m and length $2A$ situated (at equilibrium) along the x_2-axis. (The bar's moment of inertia around $\hat{x}_1$ is $I_1 = ⅓\, mA^2$ and it is thin enough to have zero moment of inertia around $\hat{x}_2$.) Find the new eigenfrequencies of the system and show that the degeneracy of the system is gone.

---

[2] (a) $\mathbf{M} = \dfrac{M}{3}\begin{pmatrix} 3 & 0 & 0 \\ 0 & A^2 & 0 \\ 0 & 0 & B^2 \end{pmatrix}$, $\mathbf{K} = 4k\begin{pmatrix} 1 & 0 & 0 \\ 0 & A^2 & 0 \\ 0 & 0 & B^2 \end{pmatrix}$, $\rightarrow \omega_{\text{S, F1, F2}} = 2\sqrt{\dfrac{k}{M}},\ 2\sqrt{\dfrac{3k}{M}},\ 2\sqrt{\dfrac{3k}{M}}$

(b) $\hat{a}_{\text{S, F1, F2}} = \dfrac{1}{\sqrt{M}}\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix},\ \sqrt{\dfrac{3}{MA^2}}\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix},\ \sqrt{\dfrac{3}{MB^2}}\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ (c) $\begin{aligned} \xi_S(t) &= \sqrt{M}\,x_3(t) = \sqrt{M}\,x_{30}\cos(\omega_S t) \\ \xi_{F1}(t) &= \sqrt{\tfrac{1}{3}MA^2}\,\theta(t) = \sqrt{\tfrac{1}{3}MA^2}\,\theta_0\cos(\omega_F t) \\ \xi_{F2}(t) &= \sqrt{\tfrac{1}{3}MB^2}\,\phi(t) = \sqrt{\tfrac{1}{3}MB^2}\,\phi_0\cos(\omega_F t) \end{aligned}$ (d) $\omega_{F1} = 2\sqrt{\dfrac{3k}{M+m}}$, others unchanged