Phys 326 Discussion 5 – Two-Body Central Force Problems

We saw that a 2-body central force system can be reduced to an effective 1-body problem by a sensible change of variables: from the individual position vectors $\vec{r}_1$, $\vec{r}_2$ to the relative position vector $\vec{r} \equiv \vec{r}_1 - \vec{r}_2$ and the CM position vector $\vec{R} = (m_1\vec{r}_1 + m_2\vec{r}_2)/M$. You then switch to the CM frame, i.e. where $\vec{R} = 0$ and $\dot{\vec{R}} = 0$. As there is no external force on this system, the CM frame will be inertial, so we can switch to it without concern.

The Result: the Lagrangian for the system is $\frac{1}{2}\mu\left|\dot{\vec{r}}\right|^2 - U(r)$ where $\mu = m_1 m_2/(m_1 + m_2)$ is the reduced mass .

Key Point: This is the Lagrangian of a **single particle** → of a mass μ moving in a central potential $U(r)$.
We can solve the entire 2-body problem by treating it as a 1-body system!

- **Coordinates** : $\vec{r} \equiv \vec{r}_1 - \vec{r}_2$, $M\vec{R} = m_1\vec{r}_1 + m_2\vec{r}_2$ .... $\vec{r}_1 = \vec{R} + \frac{m_2}{M}\vec{r}$, $\vec{r}_2 = \vec{R} - \frac{m_1}{M}\vec{r}$

Problem 1 : The Reduced Mass Must Be Derived *Hints*[1]

(a) You *need* to derive the reduced mass. Really. The transition from the [mass 1 + mass 2] system to the equivalent [mass μ + fixed central force field] system is so magical it's oddly dangerous: it will be "obvious" while in short term memory, then you'll forget it and wonder what the reduced mass was ever for …
Here's what it's for: replacing an actual 2-body system with a fake but equivalent 1-body system. You *need* to know how the 2-body to 1-body transition happens. So go for it ➔ derive $\mu = m_1 m_2/(m_1 + m_2)$ from scratch.

(b) So the system's Lagrangian is $\mathcal{L} = \frac{1}{2}\mu\left|\dot{\vec{r}}^2\right| - U(r)$. We now re-imagine the system : as some fake fixed source of the potential $U(r)$ sitting at the origin with our fake mass μ whizzing around it. The motion of this mass will be confined to a plane (we'll discuss that in lecture if it's not obvious), so it can be described by two degrees of freedom. The polar coordinates r and ϕ are perfect, giving $\mathcal{L} = \frac{1}{2}\mu\left(\dot{r}^2 + r^2\dot{\phi}^2\right) - U(r)$. We now need two equations of motion. We can get two second-order EOMs from the Lagrange equations … or we can get two first-order EOMs if we can think of two conserved quantities. There ARE two conserved quantities. Figure out what they are, and write an expression for each in terms of the reduced mass μ, the coordinates $r, \dot{r}, \phi, \dot{\phi}$, and the central potential $U(r)$.

(c) We have a complete set of EOMs, both first-order. We can call them the "E-equation" and the "L-equation". As it happens, it is also useful to have one second-order equation around: the radial force equation. Obtain this via $\partial\mathcal{L}/\partial r = d(\partial\mathcal{L}/\partial\dot{r})/dt$. (You could also take the time-derivative of the E-equation.) You will get the form: $\mu\ddot{r} =$ <term1> + <term2> … do you recognize both terms on the right-hand side?

(d) Use the L-equation to get rid of $\dot{\phi}$ in the E- and Force equations. This separates the problem into
- radial motion $r(t)$ → determined by E-equation or Force equation, for a given value of L
- angular motion $\phi(t)$ → determined by L-equation once radial motion is known

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[1] (a) Hints: Write the system's 2-particle Lagrangian for a central potential U between the two masses … switch coordinates to $\vec{R}$ and $\vec{r}$ … … convince yourself you can set $\vec{R} = 0$ … … rewrite the Lagrangian and stare … μ will be revealed. ☺
(b) Hints: Conserved quantities are energy E and angular momentum L … get L from the fact that ϕ is cyclic (or build $\vec{r} \times \vec{p}$) … get E from $T+U$ (or build the Hamiltonian, which is just $T+U$) … Answers: $L = \mu r^2\dot{\phi}$ and $E = \frac{1}{2}\mu\left(\dot{r}^2 + r^2\dot{\phi}^2\right) + U$.
(c) $\mu\ddot{r} = -U'(r) + \mu r\dot{\phi}^2 = F(r) + F_{CF}$ = sum of the central force and the **centrifugal** pseudo-force (F_{CF} is there because you have $\ddot{r}$ on the left-hand side, and that is NOT the radial component of acceleration in polar coordinates. F_{CF} is the missing acceleration term from $a_r = \ddot{r} - r\dot{\phi}^2$ moved over to the force side of the equation.) (d) see next page

All done! Here are all the equations we will use to study to behavior of 2-body central force systems:

- **L-Equation** : $\dot{\phi} = \dfrac{L}{\mu r^2}$
- **E-Equation** : $E = \dfrac{1}{2}\mu \dot{r}^2 + U + \dfrac{L^2}{2\mu r^2}$
- **Reduced Mass** : $\mu = \dfrac{m_1 m_2}{M}$
- **Force** Equation : $\mu \ddot{r} = F(r) + F_{\text{cf}}(r)$
- Effective Potential : $U^* \equiv U + U_{\text{cf}}$
- **Centrifugal** force & PE : $\vec{F}_{\text{cf}} = \dfrac{L^2}{\mu r^3}\hat{r}$, $U_{\text{cf}} = \dfrac{L^2}{2\mu r^2}$

Problem 2 : Asteroid incoming! *Checkpoints* [2]

An asteroid of mass m wanders into our solar system, heading toward the Sun, which has mass $M_\odot$. When it is very far away, the comet's speed squared is $v_0^2 = 4GM_\odot / 3b$ where b is the **impact parameter** of its trajectory toward the Sun. The definition of an impact parameter is this: if you have one object (call it the beam particle) approaching another (call it the target particle) in a straight line, the impact parameter is the minimum distance of approach that the beam will come to the target. Makes sense? Impact parameters are used all the time in scattering problems … which this problem is, in fact. ☺ So: when our asteroid is far away, it has the speed v_0 given above and it is on a straight-line course that, *if left undeflected*, would have it pass by the Sun at a minimum distance of approach of b. But the sun *is* there, of course, and it affects the asteroid's trajectory when it gets close enough.

(a) Throughout this problem, you can assume that the asteroid is so much lighter than the sun ($m \ll M_\odot$) that you can approximate μ as simply equal to one of the two masses. Which one?

(b) The E-equation can be written $E = \frac{1}{2}\mu \dot{r}^2 + U^*(r)$ where the **effective potential energy** U^* has two terms: $U(r)$ from the central force plus the term $L^2 / 2\mu r^2$, which can be termed "centrifugal potential". Make a quick sketch of the effective potential $U^* = U + U_{\text{cf}}$ seen by the asteroid as a function of r. This plot is absolutely crucial to understanding what the asteroid will do!

As we will see, the r coordinate will always be either trapped between some minimum value r_{min} and some maximum value r_{max} (**bounded orbit**), or else it will come in from infinity, reach some value r_{min}, then head back out to infinity (**unbounded orbit**). The type of orbit you get depends on the system's energy E. The radial "turning points" at r_{min} and/or r_{max} are such key features of orbits that they have special names:

- **apse** ≡ point on orbit where $r = r_{\text{min}}$ or r_{max}
- **apsidal distances** ≡ r_{min} and/or r_{max}
- **apsidal angle** ≡ angle between successive apses
- **peri-** ≡ means "near", refers to the r_{min} apse
- **ap(o)-** ≡ means "far", refers to the r_{max} apse
- **perigee / apogee** ≡ apses around <u>Earth</u>
- **perihelion, aphelion** ≡ apses around <u>Sun</u>

(c) Calculate the **perihelion distance** reached by the asteroid, i.e. the closest distance of approach to the Sun.

(d) Calculate the asteroid's speed at perihelion.

(e) Make a sketch of the asteroid's path through space as it passes the Sun.

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[2] (a) the asteroid (c) $b/2$ (d) $2v_0$. Did you get zero? You're correct … for one of the *components* of velocity. There's another one!