Phys 326 Discussion 6 – Central Force Orbits

Here is our growing collection of useful formulae concerning the two-body central force problem. Remember from our study of particle collections in 325: unscripted CAPITAL letters denote TOTALs for the system if they are additive quantities (e.g. $M = m_1 + m_2$) or CM properties if they are not (e.g. $\vec{R}$ = CM position).

- **Coordinates** : $\vec{r} \equiv \vec{r}_1 - \vec{r}_2$, $M\vec{R} = m_1\vec{r}_1 + m_2\vec{r}_2$ .... $\vec{r}_1 = \vec{R} + \dfrac{m_2}{M}\vec{r}$, $\vec{r}_2 = \vec{R} - \dfrac{m_1}{M}\vec{r}$
- **L Equation** : $\dot{\phi} = \dfrac{L}{\mu r^2}$
- **E Equation** : $E = \dfrac{1}{2}\mu\dot{r}^2 + \left(U + \dfrac{L^2}{2\mu r^2}\right) = \dfrac{1}{2}\mu\dot{r}^2 + U^*$
- **Reduced Mass** : $\mu = \dfrac{m_1 m_2}{M}$
- **Force Equation** : $\mu\ddot{r} = F(r) + \dfrac{L^2}{\mu r^3} = F + F_{cf}$
- **Path Equation** : with $u(\phi) \equiv \dfrac{1}{r(\phi)}$ $\rightarrow$ $u'' + u = -\dfrac{\mu F(1/u)}{L^2 u^2}$ and $u' = -\dfrac{\mu\dot{r}}{L}$

The new entry in our collection compared with last week's discussion is the **Path Equation**.

- The E or Force equation gives us the radial motion $r(t)$ as a function of time, for a given ang. momentum L.
- The L equation gives us the angular motion $\phi(t)$ as a function of time, once we know $r(t)$.
- The Path equation bypasses time entirely and gives us $r(\phi)$ $\rightarrow$ the **shape** of the orbit in polar coordinates.

The one trick you need to remember about the path equation is to change variables from r to $u \equiv 1/r$.
It just turns out that paths are *much* easier to calculate in the form $u(\phi)$ than in the form $r(\phi)$.

Problem 1 : Comet Repelled by a Mysterious Object *Checkpoints* [1]

A mysterious interstellar object O exerts a repulsive inverse cube force $\vec{F} = (m\gamma / r^3)\hat{r}$ on any mass m in its vicinity. A comet of mass m starts a great distance away, and moves toward O with speed $v_0 = 3\sqrt{\gamma} / (4b_0)$ along a straight line whose perpendicular distance from O is b_0. You may assume that the comet's mass m is much less than that of the mysterious object O (so μ becomes very simple ☺).

(a) Use the **path equation** to obtain the shape of the orbit, $u(\phi)$. For definiteness, define $\phi = 0$ to correspond to the comet's initial position when it is infinitely far away from the mysterious object.

(b) Calculate the apsidal angle of the comet's trajectory = the net angular interval $\Delta\phi$ through which the particle moves from its original position at infinity to its one apse. To do this, plot the function $u(\phi)$ you just found, then mentally take the inverse $1/u$ of your sketch to plot r vs ϕ. Now recall from lecture: the L equation tells us that the angle ϕ is always increasing (or always decreasing) with time. Your $r(\phi)$ plot thus also shows us the time-evolution of the comet's r coordinate. Since the comet came in from infinity, it is in an unbounded orbit … what you should find is that r decreases from infinity to a minimum value (periapsis) then back out to infinity. The apsidal angle you are looking for is the angle $\Delta\phi$ between either of those infinities and the periapsis point of closest approach.

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[1] (a) $u(\phi) = (3/5b_0)\sin(5\phi/3)$ (b) 54° (d) $5b_0/3$ (e) checkpoint: $T = (u'^2 + u^2)L^2/m^2$ … final: $T(\phi) = v_0^2\left[1 - \dfrac{16}{25}\sin^2\left(\dfrac{5\phi}{3}\right)\right]$ (f) $U(r) = m\gamma/(2r^2)$ (g) checkpoint: $U = \gamma u^2$ ... final: $U(\phi) = \dfrac{16}{25}v_0^2\sin^2\left(\dfrac{5\phi}{3}\right)$ (h) v_0^2 = total E (i) $U_{cf} = \frac{1}{2}m(r\dot{\phi})^2$ = part of T!

(c) Sketch the comet's path, indicating the apsidal angle $\Delta\phi$ on your sketch. (It should appear in two places).

(d) Calculate the periapsis distance of the comet's trajectory = its distance of closest approach to O.

(e) Calculate and plot the comet's kinetic energy $T = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\phi}^2)$ as a function of ϕ over its trajectory. Take m=2 for convenience; your $T(\phi)$ expression will then only involve v_0^2, ϕ, and numerical values.

(f) Figure out the formula for the potential energy $U(r)$ that corresponds to the force $\vec{F} = (m\gamma / r^3)\hat{r}$ exerted by the mystery object O.

(g) Now calculate and plot the comet's potential energy $U(\phi)$ as a function of angle. To compare with your $T(\phi)$ result from part (e), again use m=2 and express your result in terms of v_0^2, ϕ, and numerical values.

(h) Add $T(\phi)+U(\phi)$. Are you happy with your result? What does it correspond to?

(i) If all went well, you found that $T(\phi)+U(\phi)$ is just the total, constant energy of the comet. Great. But wait … the potential energy you used was the potential energy due to the central force … what happened to the centrifugal potential energy $U_{\mathrm{cf}} \equiv L^2 / 2\mu r^2$? Where did it go? If you're not sure, calculate $U_{\mathrm{cf}}(\phi)$ and see if you can find it somewhere within your total $E = T(\phi) + U(\phi)$ expression.

Problem 2 : Kepler's Second Law *Incremental Hints*[2]

Kepler's three famous laws are all observational. Unlike Newton's Laws, they don't explain the theory *behind* the motion of celestial bodies, rather they *describe* key features of that motion, with a level of accuracy and insight that was unprecedented in the 17th century. In the 21st century, we can derive the results from our underlying theory of mechanics and gravitation. Kepler's second law is already within your grasp. It says: "For each of the planets, the straight line connecting the planet to the Sun sweeps out equal areas in equal times." You can derive this using a feature incorporated into several of our above equations: that the angular momentum L of the Sun-planet system is conserved. Amazingly, you don't need to know anything about the shape of the orbit other than it's in a single plane (which is also a consequence of the angular momentum vector $\vec{L}$ being conserved). Go ahead and derive the relation $dA / dt = L / 2\mu$ = constant for a particle of mass μ orbiting around a fixed origin. A long chain of incremental hints is provided in the checkpoint, see if you can find the solution without using all of them. ☺

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[2] Hints: SKETCH a segment of a generic orbit (it can have any shape, it only has to be smooth) and draw the little wedge of area dA that is swept out when you change angle by a little bit $d\phi$ … … for infinitesimally small $d\phi$, the little wedge dA swept out can be approximated as a *triangle* with straight sides … … write vectors for two sides of the triangle, remembering that we want the amount dA that's swept out in a tiny *time* interval dt (so we can construct dA/dt at the end) … … one of the vectors is $\vec{r}$ = the position vector pointing to the spot where we began our little motion … … the vector along the orbit involves the vector $\vec{v}$ and something to give it the right units … … it is $\vec{v}\,dt$ … … the little triangle can be turned into a parallelogram of area 2 dA … … what simple formula gives you the area of a parallelogram in terms of the vectors describing two of its sides? … … *cross-product* … … what is the formula for angular momentum? … … $\vec{L} = \vec{r} \times \mu\vec{v}$ … … rearrange to get dA / dt. ☺