Phys 326 Discussion 7 – Captured Paths and Transfer Orbits

• **Coordinates & Reduced Mass** : $\vec{r}_1 = \vec{R} + \frac{m_2}{M}\vec{r}$, $\quad \vec{r}_2 = \vec{R} - \frac{m_1}{M}\vec{r}$, $\quad \mu = \frac{m_1 m_2}{M}$

• **Centrifugal** force & PE : $\vec{F}_{cf} = \frac{L^2}{\mu r^3}\hat{r}$, $\quad U_{cf} = \frac{L^2}{2\mu r^2}$, $\quad$ effective $U^* = U + U_{\text{cf}}$

• **Angular EOM** : $\dot{\phi} = \frac{L}{\mu r^2}$ $\qquad$ • **Radial EOMs** : $\mu\ddot{r} = F(r) + F_{\text{cf}}(r)$, $\quad E = T + U(r) = \frac{1}{2}\mu\dot{r}^2 + U_{\text{cf}}(r) + U(r)$

• **Path Equation** : $u(\phi) \equiv 1/r(\phi) \;\rightarrow\; u'' + u = -\frac{\mu F(1/u)}{L^2 u^2}$ and $u' = -\frac{\mu\dot{r}}{L}$

• **Ellipse / Hyperbola** with (r,ϕ) centered on a <u>focal point</u> : $\frac{1}{r} = \frac{a}{b^2}(1 + e\cos\phi)$, $\quad e = \frac{c}{a} = \frac{\sqrt{a^2 \mp b^2}}{a}$

• **Kepler Orbits** for $F = -\frac{\gamma}{r^2}$: $r(\phi) = \frac{r_0}{1 + e\cos\phi}$ with $r_0 = \frac{L^2}{\mu\gamma} = \frac{b^2}{a}$, $\quad E = \mp\frac{\gamma}{2a} = \frac{\gamma(e^2-1)}{2r_0}$, $\quad \tau^2 = \frac{4\pi^2\mu}{\gamma}a^3$

Also, the $r(\phi)$ formula in the last line immediately gives $r_0 = \frac{2\,r_{\min}r_{\max}}{r_{\min} + r_{\max}}$ and $e = \frac{r_{\max} - r_{\min}}{r_{\max} + r_{\min}}$.

Problem 1 : A Captured Path *Checkpoints for* [1] & [2]

In lecture we introduced the <u>capture cross-section</u>, $\sigma = \pi b_{\max}^2$: particles fired with a given initial velocity $\vec{v}_0$ at a potential well are "captured" by it when their impact parameter b is below some value $b_{\max}$. "Captured" means the particle cannot return to $r = \infty$ (cannot "escape" the potential well). Many of you wondered what exactly happens to such a captured particle … good question! Let's find out, using the attractive "Death Star" force from lecture, $F = -k/r^3$, which corresponds to the potential $U = -k/2r^2$. We will send in particles with initial velocity v_0, impact parameter b, and mass $m << M_{\text{DeathStar}}$.

(a) Plot $U^*(r)$. You will see that it can take two different shapes, depending on the relative size of the force constant k and L^2/m for the incoming particle. Using the fact that L is related to b, find the capture condition on b, and from that get $\sigma = \pi b_{\max}^2$.

(b) Use the path equation to determine the path $r(\phi)$ for captured particles of mass m. You will quickly find it convenient to define the symbol $\alpha \equiv \sqrt{(mk/L^2) - 1}$.

(c) Sketch the path! Important: $\dot{\phi} = L/\mu r^2$ tells us that <u>the angle is constantly increasing with time</u>; it will keep growing until r reaches some limit like $r = \infty$ or $r = 0$. What shape is the path? Are the particles actually "captured" by the Death Star?

(d) Calculate the speed components $v_r = \dot{r}$ and $v_\phi = r\dot{\phi}$ as functions of ϕ. What happens to them in the limit $\phi \rightarrow \infty$? (We are using ϕ as a substitute for time; we could solve for time, but it is messy.) Think of one or two reasons why the limit you obtained will not occur in practice, despite your calculation.

---

[1] (a) $\sigma = \pi(k/mv_0^2)$ (b) $r(\phi) = \alpha b/\sinh(\alpha\phi)$ (c) path = inward spiral (d) $v_r = -v_0\cosh(\alpha\phi)$, $v_\phi = (v_0/\alpha)\sinh(\alpha\phi)$

[2] (d) Hint: the transfer time is half the period of the transfer orbit. Answer: $T = \pi\sqrt{(R_1 + R_3)^3/(8GM_\odot)} = 2.7$ years.

(e) $(v_1, v_{2L}, v_{2R}, v_3) = (30, 39, 7.4, 13)$ km/sec. (f) It doesn't matter *what* radial speed you use: if you have to *stop* the angular velocity v_1 at L and *create* angular velocity v_3 at R, your total Δv will be at least $v_1 + v_3 = 43$ km/s > Hohmann's 15 km/s !

Problem 2 : Hohmann Transfer to Jupiter *Checkpoints on previous page*

The idea of **Hohmann Transfer Orbits** dates from 1925 when Walter Hohmann found the most cost-effective procedure for transferring a spacecraft from one circular orbit to another circular orbit in the same plane.

(a) Here are the steps involved; draw them as your read so you can visualize the procedure.

1. Draw the Sun, and draw two circular orbits around it. Assign the starting orbit a radius R_1 and assign the ending orbit a radius $R_3 > R_1$. We will imagine that orbit 1 is the Earth's orbit, while orbit 3 is that of Jupiter. Our goal is to get a spaceship from Earth-orbit to Jupiter-orbit (so it can take pretty pictures ☺).
2. Hohmann's technique to get a ship from orbit 1 to orbit 3 is to put the ship into a temporary **transfer orbit**, which we call orbit 2. Draw in this transfer orbit as a dashed line using Hohmann's specification: the transfer orbit is an ellipse that is tangential with both orbit 1 and orbit 3.
3. The ship will briefly fire its rockets at two points: at the point L=Launch where we switch the ship from orbit 1 to orbit 2, and at the point R=Rendezvous where switch from orbit 2 to orbit 3. Label these two points L and R on your drawing. (They are the points where the circular orbits touch the transfer orbit.)

Does everything make sense? If your drawing is correct you will see that the transfer orbit has perihelion distance R_1 and aphelion distance R_3. Also, you can see that the two rocket firings will apply impulses that are tangential to the ship's current orbit. This avoidance of any radial velocity changes is at the heart of the cost-saving of the Hohmann procedure. It is not the fastest way to get from one orbit to another; rather, it uses the least amount of fuel, and getting *anything* up into space is extremely expensive!

(b) We need very little information to get all the numbers for this problem. We need the distance from the Sun to Jupiter, which is 5.2 A.U., where an A.U. (Astronomical Unit) is the Sun-Earth distance. Almost everything else can be obtained from well-known quantities about the Earth's orbit (well-known because we live here. ☺) For example, the nasty constants G, $M_\odot$, and $R_1 = 1$ A.U. will appear frequently in our calculations, but in **convenient combinations**. Specifically, $(GM_\odot / 1\,\text{A.U.})$ turns out to be a very pretty quantity: 9×10^8 (m/s)^2. Show that, since the Earth's orbit is nearly circular, the velocity of the Earth around the Sun is the square root of this pretty quantity: $v_1 = \sqrt{GM_\odot / R_1} = 30\ \text{km/s}$.

(c) Another convenient combination of G, $M_\odot$, and $R_1 = 1$ A.U. can be obtained from the 1-year period of the Earth around the Sun. Show that $\sqrt{GM_\odot / R_1^3} = 2\pi / \text{year}$. Combined with (b), we also get 30 km/s = 2π AU/yr.

(d) Now calculate the total time T it will take to get from Launch to Rendezvous. A hint is in the footnote.

(e) Calculate the four speeds v_1, v_{2L}, v_{2R}, and v_3 involved in the story. In case the notation is unclear: v_1 is the ship's speed in orbit 1; v_{2L} is the speed right after the ship fires its rockets at point L (which puts it onto orbit 2); v_{2R} is the speed on orbit 2 right *before* the rockets are fired for a second time, at point R; and v_3 is the ship's final speed in orbit 3.

(f) Does Hohmann Transfer really optimize cost by minimizing the amount of fuel burned? As we showed in 325, the amount of fuel burned is proportional to the sum of velocity changes $|\Delta v|$. From part (e), you found $|\Delta v_L| + |\Delta v_R| \approx 15$ km/s for Hohmann, so experiment and see if you can do better! Try a direct radial trajectory: fire your rockets at L=Launch to give the ship zero angular speed and a radial speed that will reach Jupiter's orbit in the same time $T = 2.7$ years that Hohmann requires. Once you reach Jupiter's orbit, remember that you have another velocity change to make: you must stop your radial velocity and impart an angular velocity matching Jupiter's orbit. See how much "cost" $|\Delta v_L| + |\Delta v_R|$ your plan requires!