Phys 326 Discussion 8 – Rutherford Scattering & Repulsive Kepler Trajectories

Our Kepler formula-set is now updated to include two things: (1) the possibility of repulsive $1/r^2$ forces with negative force-constants γ (2) relations needed for scattering problems, namely formulae for the scattering angle θ and impact parameter b for unbounded Kepler orbits as well as general cross-section formulae.

• **Coordinates & Reduced Mass** : $\vec{r}_1 = \vec{R} + \frac{m_2}{M}\vec{r}$, $\quad \vec{r}_2 = \vec{R} - \frac{m_1}{M}\vec{r}$, $\quad \mu = \frac{m_1 m_2}{M}$

• **Centrifugal** force & PE : $\vec{F}_{cf} = \frac{L^2}{\mu r^3}\hat{r}$, $\quad U_{cf} = \frac{L^2}{2\mu r^2}$, $\quad$ effective $U^* = U + U_{\text{cf}}$

• **Angular EOM** : $\dot{\phi} = \frac{L}{\mu r^2}$ $\quad$ • **Radial EOMs** : $\mu\ddot{r} = F(r) + F_{\text{cf}}(r)$, $\quad E = T + U(r) = \frac{1}{2}\mu\dot{r}^2 + U_{\text{cf}}(r) + U(r)$

• **Path Equation** : $u(\phi) \equiv 1/r(\phi) \;\rightarrow\; u'' + u = -\frac{\mu F(1/u)}{L^2 u^2}$ and $u' = -\frac{\mu\dot{r}}{L}$

• **Conics** : With (r,ϕ) centered on a focal point and E ≡ Ellipse, H ≡ Hyperbola

$$\frac{1}{r} = \frac{a}{b^2}(\pm 1 + e\cos\phi) \text{ with } \begin{cases} +\text{: E or H-near-branch} \\ -\text{: H-far-branch} \end{cases} , \qquad e = \frac{c}{a} = \frac{\sqrt{a^2 \mp b^2}}{a} \text{ with } \begin{cases} -\text{: E} \\ +\text{: H} \end{cases}$$

• **Kepler Orbits** $F = -\frac{\gamma}{r^2}$: $r(\phi) = \frac{r_0}{\text{sgn}[\gamma] + e\cos\phi}$ with $r_0 = \frac{L^2}{\mu|\gamma|} = \frac{b^2}{a} = a\left|1-e^2\right|$, $\quad E = \mp\frac{|\gamma|}{2a} = \frac{|\gamma|(e^2-1)}{2r_0}$

Bounded orbits: $\quad \tau^2 = \frac{4\pi^2\mu}{\gamma}a^3$, $\quad r_0 = \frac{2\,r_{\min}r_{\max}}{r_{\min}+r_{\max}}$, $\quad e = \frac{r_{\max}-r_{\min}}{r_{\max}+r_{\min}}$

Unbounded orbits: $\quad$ scattering angle $\theta = \pi - 2\alpha$ with $\tan\alpha = \frac{b}{a}$, $\quad$ impact parameter b = semi-minor axis b ☺

• **XSec** : $d\Omega \equiv \frac{dA}{r^2} = \begin{cases} \sin\theta\, d\theta\, d\phi \\ d\theta_x\, d\theta_y \end{cases}$, $\quad \frac{d\sigma}{d\Omega} = \frac{b}{\sin\theta}\left|\frac{db}{d\theta}\right|$ with θ = scattering angle $\quad$ • **Lumi**: $\mathcal{L} = n_A N_e$, $\quad \frac{dN_{\text{ev}}}{dt} = \mathcal{L}\sigma$

Problem 1 : The Rutherford Cross Section

We have all the tools we need to derive the most famous and most commonly-used cross section in the world: the **Rutherford XSec** $d\sigma/d\Omega$ for the non-relativistic scattering of two charged particles. Ernest Rutherford used this calculation to analyze the 1911 scattering experiment of Geiger and Marsden and deduce that the positive charge in the atom is *not* smeared uniformly within the atom but *concentrated in a tiny volume*. This was the discovery of the atomic nucleus. On to our derivation! We give the beam particle a charge q and the target particle a charge Q; the force between them is then $F = kqQ/r^2$. We also assume that the target particle's mass is so much greater than the beam particle's mass ($M \gg m$) that the target can be treated as fixed.

You have all the tools you need to show that this famous cross-section is $\frac{d\sigma}{d\Omega} = \left[\frac{kQq}{4E\sin^2(\theta/2)}\right]^2$.

If you need them, steps are in the footnote on the next page. Also you will find these trig relations helpful:

$$\tan\left(\frac{\pi}{2} - \theta\right) = \cot(\theta) , \qquad \frac{d}{d\theta}\left[\cot(\theta/2)\right] = -\frac{1}{2\sin^2(\theta/2)} , \qquad \sin\theta = 2\sin(\theta/2)\cos(\theta/2)$$

Steps for Problem 1 [1]
Checkpoints [2]

Problem 2 : Geiger and Marsden 1911

Geiger and Marsden's historic experiment scattered α particles of kinetic energy $E = 7.7$ MeV from a gold foil. How close did their beam get to the target gold nuclei?

(a) Obtain a formula for $r_{\min}$ = distance of closest approach as a function of a and b for a repulsive $1/r^2$ force.

(b) The closest approach depends on impact parameter. For what impact parameter will you get as close as possible to the target?

(c) Calculate the smallest-possible distance of closest approach, $r_{\min}$ for $b = 0$, and figure out its value for the Geiger and Marsden experiment. You will need to know that Z = 2 for α particles (which are just ^{4}He nuclei) and Z = 79 for gold to obtain the beam and target charges, and in case you forget, $k = 9E9$ in SI units. ☺
☞ **ZERO-over-ZERO Problem?** If you just set $b = 0$ in your $r_{\min}$ formula from (a), you will get 0/0. You have not made an error … can you figure out how to handle that little 0/0 problem? If not, see the footnote!

☞ **What does your value tell you?** When Rutherford analyzed Geiger and Marsden's measurement of the scattering cross section versus scattering angle θ, he found that it *exactly matched* the formula you derived in problem 1. That formula assumes the electric field is kQ/r^2 at every point in space that the beam can reach, i.e. that the target's charge Q is *completely concentrated inside a volume that the beam has not penetrated*. Do you see the point? If the charge Q was spread out over a broad region of space, the E field would be $kQ_{\text{enc}}(r)/r^2$ where $Q_{\text{enc}}(r)$ is the charge enclosed out to radius r. Well, you just calculated that the 1911 experiment was able to get within 60 fm of the target's center, and they saw no deviation from the Rutherford formula for scattering from a point charge. They knew that the radius of the atom was around 10^{-10} m which is *way* bigger than 60 fm!

Problem 3 : Scattering Relations for the Far-Side Hyperbola Branch

Encouragement [3]

In lecture, we related the standard parameters of scattering experiments, namely scattering angle θ & impact parameter b, to the geometric parameters of hyperbolae, namely semi-major axis a & semi-minor axis b. Our derivation was nearly all geometry … but we only inspected the *near* branch of a hyperbola. We must ensure that our relations also apply to the far branch of a hyperbola, since that's the branch we get when repulsive $1/r^2$ forces are at play (as in the Rutherford cross section!). Set up an (x,y) coordinate system, place a focal point at $(x,y) = (c,0)$ as usual, then draw in a far branch hyperbola relative to this focal point. As we did in lecture, identify the incoming beam, the target, and the scattered particle trajectory on your sketch. Finally, use conic-section relations to show that the scattering angle is still $\theta = \pi - 2\alpha$ where $\tan\alpha = b/a$ and the impact parameter b is the same as the semi-minor axis b $\rightarrow$ exactly the same as for the near-branch hyperbola you obtain from attractive $1/r^2$ forces.

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[1] Here are the steps to take in your derivation. **Step 1: strategize** (A) Looking at our $d\sigma / d\Omega$ formula, we see that we need to relate impact parameter b to scattering angle θ $\rightarrow$ that's our main goal. (B) Looking at the answer we're trying to obtain, we see that the given parameters — the "things we know" — are going to be E, θ, and of course $kQq = |\gamma|$. So our "strategy box" looks like this: we know E, θ, $|\gamma|$ and we want b. **Step 2: get *b* in terms of *θ*** We have a $1/r^2$ force here, so we are in the world of Kepler orbits $\rightarrow$ go through that rich section of the formula sheet looking for the relations you need to determine what you want = b in terms of what you know = E, θ, $|\gamma|$. Result: $b = (|\gamma| / 2E)\cot(\theta/2)$. **Step 3: get dσ/dΩ** Now it's just plug-and-play using our differential cross section formula and the trig relations provided.

[2] (a) $r_{\min} = \dfrac{b^2/a}{-1+\sqrt{1+b^2/a^2}}$ (b) impact parameter $b = 0$ gives closest possible distance-of-approach (c) 29.5 fm. To deal with the zero/zero problem, first **approximate** your $r_{\min}$ formula for very small b, *then* take the limit $b \rightarrow 0$; you will get $r_{\min} = 2a = |\gamma| / E$.

[3] It's all sketching and geometry, enjoy! And you are a wonderful person!