Phys 326 Discussion 9 – Symmetries and Principal Axes

Today we will study some important special cases that can greatly simplify our work with inertia tensors: when an object has certain **symmetries**, we can guess the principal axes in advance. This is hugely valuable! If we know the principal axes (**PA**s) in advance, we can set up a coordinate system that follows them and *then* calculate the inertia tensor. As you will prove in the first problem, the inertia tensor is diagonal in this case.

- $I_{ij} = \int dm\ (\delta_{ij} r^2 - r_i r_j) \to \mathbf{I} = \int dm \begin{pmatrix} y^2+z^2 & -xy & -xz \\ \cdot & z^2+x^2 & -zy \\ \cdot & \cdot & x^2+y^2 \end{pmatrix}$
- $\vec{L}^{(B)} = \mathbf{I}^{(B)}\vec{\omega} \quad \forall$ body-fixed ref. point B
- Principal Axes $\hat{e}$ (definition): $\mathbf{I}\hat{e} = \lambda\hat{e}$
- Kinetic Energy: $T = \frac{1}{2}\vec{\omega}\cdot\vec{L} = \frac{1}{2}\vec{\omega}^T\mathbf{I}\vec{\omega}$
- Parallel-Axis Theorem: $\mathbf{I}^{(B)} = \mathbf{I}^{(B)}_{CM} + \mathbf{I}' = M\left(\delta_{ij}R^{(B)2} - R_i^{(B)}R_j^{(B)}\right) + \int dm\left(\delta_{ij}r'^2 - r_i' r_j'\right)$

IMPORTANT ADVICE : The great mathematician Carl Gustav Jacobi used to tell his students "*Invert, always invert!*" He meant: the solution to a problem can often be simplified by inverting the problem, i.e. by **working backwards**. For a proof, invert the problem by **disproving the converse** → hypothesize that the statement you are trying to prove is false, then show that this leads to a contradiction. This advice will be very helpful today!

Problem 1 : Diagonal I *Hints* [1],[2]

(a) Prove that if $\hat{z}$ is a principal axis of an object, then its inertia tensor must have this form: $\mathbf{I} = \begin{pmatrix} I_{xx} & I_{xy} & 0 \\ I_{xy} & I_{yy} & 0 \\ 0 & 0 & I_{zz} \end{pmatrix}$

(b) Prove that if $\hat{x}$, $\hat{y}$, and $\hat{z}$ are all principal axes of an object, then its inertia tensor **I** is a diagonal matrix.
☞ Here lies the enormous advantage of using a coordinate system that follows the principal axes!

(c) Convince yourself that the three diagonal elements of **I** are the familiar **moments of inertia** from 211 / 325. (i.e. convince yourself that the element I_{xx} is just the moment of inertia $I_{\hat{x}}$ for rotation around the x axis).
☞ If your matrix is diagonal, the eigenvalues I_{xx}, I_{yy}, & I_{zz} are the *only* elements you need; since they are ordinary moments-of-inertia, you can often look them up, which makes building **I** very simple. ☺

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[1] **Q1** (a) Follow Jacobi's advice → suppose **I** does *not* have the given form, i.e. that those zeros are non-zero ... then show that there is no way the z-axis can be a principal axis. (b) pretty obvious (c) Just look at the formula for I_{ij}

[2] **Q2** (a) Hint: pick a good coordinate system (origin at point O, lamina in xy-plane), then consider which elements of the inertia tensor are zero by staring at the integrals you must perform to evaluate them.
(b) **Strategy 1 = geometry:** Follow Jacobi's advice → suppose that $\hat{z}$ is *not* a PA … which means that $\vec{\omega} = \omega\hat{z}$ results in an angular moment $\vec{L} = I\vec{\omega}$ that is *not* be parallel to $\hat{z}$, it is some vector (L_x, L_y, L_z) with L_x and/or L_y non-zero … reflect the system across the xy-plane … the object is exactly the same as it was before, but $\vec{\omega} = -\omega\hat{z}$ and $\vec{L} = (L_x, L_y, -L_z)$ … change the sign of ω ... how does $\vec{L}$ change? … $\vec{L} = I\vec{\omega}$ ∴ all components of $\vec{L}$ change sign: $(-L_x, -L_y, L_z)$ … and now you have the exact same object you started with and the exact same $\vec{\omega} = \omega\hat{z}$ you started with, but … $\vec{L} = (-L_x, -L_y, L_z)$ is *not* the same! This is impossible → $\hat{z}$ must be a PA.
Strategy 2 = math: "Reflection symmetry across the xy-plane" means "nothing changes if you change z to $-z$" … that means (i) the object's mass density $\varrho(x,y,z)$ is an even function of z, and (ii) at any value of x and y, the object's bounds in z are even, i.e. $z_{\max}(x,y) = -z_{\min}(x,y)$ … evaluate some elements of **I** … you get the same form of **I** as in 1(a) ... which proves that the z axis is a PA.
(c) Placing a box face in the xy-plane and the origin at center of the face: $\hat{x}$ & $\hat{y}$ are PAs by reflection symmetry across yz-plane and xz-plane respectively ... $\hat{z}$ is a PA because of the fundamental **eigenvector theorem**: since **I** is real and symmetric, it must have 3 perpendicular eigenvectors; we know $\hat{x}$ & $\hat{y}$ are two of them, so $\hat{x}\times\hat{y} = \hat{z}$ must be the third.
(d) $\hat{z}$ by lamina theorem, $(\hat{x}-\hat{y})/\sqrt{2}$ by reflection symmetry, $(\hat{x}+\hat{y})/\sqrt{2}$ by eigenvector theorem.

Problem 2 : Lamina and Reflection Symmetry

(a) A **lamina** is a flat 2D object such as a thin (zero-thickness) plate of sheet metal. Consider a lamina that is rotating about a point O located in the lamina. Prove that the vector perpendicular to the plate is a PA.

(b) Consider an object that is **unchanged** by **reflection across some plane** that passes through the object. (For convenience, let's make this symmetry plane the xy-plane.) Suppose the object is rotated around a point O that lies in the symmetry plane. (For convenience, place the origin at the point O.) Show that the axis **perpendicular to the symmetry plane** (i.e. the z-axis) is a **principal axis**. Note: this theorem provides an extension of the lamina theorem to include thick flat plates of uniform density (as long as the point of rotation lies in the mid-plane of the plate).

(c) Consider a uniform rectangular slab, e.g. a box. What are the principal axes for rotation of the box around the center of one of its faces? (not the center of the box) Provide an argument for each principal axis you propose without calculating anything and sketch the result.

(d) Consider a thin flat right-triangle that lies in the xy-plane with vertices at $(x,y) = (0,0)$, $(1,0)$, and $(0,1)$. Without any calculation, figure out its principal axes for rotation around the origin.

Problem 3 : Axisymmetry *Hints & Checkpoints* [3]

(a) Consider an arbitrary rigid body with an **axis of rotational symmetry**, which we'll call $\hat{z}$. What this **axisymmetry** means is that if you rotate the object by any angle around $\hat{z}$, the object is completely unchanged. Prove that the axis of symmetry $\hat{z}$ is a principal axis.

(b) Prove that any direction perpendicular to $\hat{z}$ (e.g. $\hat{x}$ or $\hat{y}$) is *also* a principal axis.

(c) Finally, prove that the principal moments for rotation around the $\hat{x}$ and $\hat{y}$ eigenvectors are *equal* ($I_{xx} = I_{yy}$).
☞ Axisymmetric objects have *really simple* inertia tensors, with only two unique non-zero moments (I_{zz} & $I_{xx} = I_{yy}$) … as long as you are rotating around a point that lies on the symmetry axis, of course.

Problem 4 : Degenerate Eigenvalues *Checkpoints* [4]

Many inertia tensors have two or three eigenvalues that are *the same*. Such repeated eigenvalues are called **degenerate eigenvalues**. When this situation occurs, you will have some **freedom** in your choice of the corresponding eigenvectors, i.e. they will not be uniquely determined. For convenience, let's align our object so its three principal axes are $\hat{x}$, $\hat{y}$, and $\hat{z}$, and let's rotate it around the origin. Suppose that the moments of inertia I_{xx} and I_{yy} are equal (i.e. degenerate), while I_{zz} is different. Show that **any linear combination of the degenerate eigenvectors is also an eigenvector.**

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[3] **Q3** (a) Follow Jacobi's advice → Hypothesize that $\hat{z}$ is *not* a PA, then find a consequence that contradicts the symmetry of the object … recall the definition of a PA: rotation around a PA produces $\vec{L}$ parallel to $\vec{\omega}$; conversely, if you rotate around an axis that is *not* a PA, $\vec{L}$ is *not* parallel to $\vec{\omega}$ … keep following strategy 1 from question 2(b) (b,c) You're an expert now ☺

[4] **Q4** Build the inertia tensor for this object … it's diagonal … construct a random vector $\vec{\omega}$ that is a linear combination of the eigenvectors with degenerate eigenvalues … that would be $\vec{\omega} = \omega_x \hat{x} + \omega_y \hat{y}$ … calculate $\vec{L} = I\vec{\omega}$ and show that $\vec{L}$ is parallel to $\vec{\omega}$.