Phys 326 Discussion 10 – Rotational Trajectories; Euler's Equations

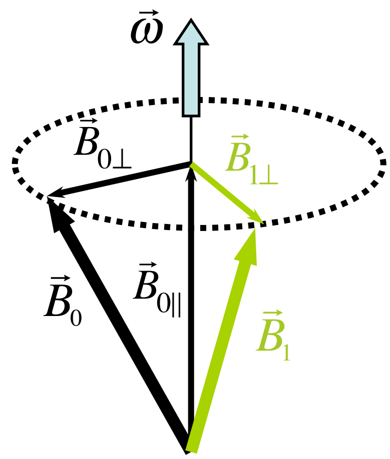
Problem 1 : Time-Dependence of Vectors Under Constant Rotation $\vec{\omega}$ *Checkpoints* [1]

In Phys 325 we proved this crucial relation:

$$\frac{d\vec{B}}{dt} = \vec{\omega}\times\vec{B} \quad \text{for any constant vector } \vec{B} \text{ that is fixed in a body rotating with angular velocity } \vec{\omega}.$$

You have used this result many times. A particularly common use is taking $\vec{B}$ to be the position-vector $\vec{r}$ of a fixed point in the rotating body → the point's velocity in the lab frame is then $\vec{v} = \dot{\vec{r}} = \vec{\omega}\times\vec{r}$. Alternatively the "rotating body" might be just a point mass whizzing around some axis with angular velocity $\vec{\omega}$: mentally attach the mass to any point O on the axis of rotation using a massless rod, and you have created a rigid body. If $\vec{r}$ is the position of the mass relative to O, it's a body-fixed vector, so $\vec{\omega}\times\vec{r}$ is the mass's velocity.

We've used $\dot{\vec{B}} = \vec{\omega}\times\vec{B}$ plenty of times, but we haven't yet *solved* it. By "solved", I mean that if you know $\vec{\omega}$, $\dot{\vec{B}} = \vec{\omega}\times\vec{B}$ is a first-order differential equation that can be solved to obtain the <u>components $B_i(t)$</u>, i.e. the explicit time-dependence of the full vector $\vec{B}$ in the lab frame. (You will need to do this on the last problem of the homework due this week, for example.) The solution is easy to find using geometric thinking in the case of **constant angular velocity** $\vec{\omega}$. This is a very common case, so let's study it!



Here is the super-common situation we want to solve: we have some vector $\vec{B}$ that has these two properties:

- it is rotating with constant angular velocity $\vec{\omega}$
- at time t=0 it is $\vec{B}_0$

So our given quantities are two constant vectors $\vec{\omega}$ and $\vec{B}_0$; what we want to figure out is the **time-dependence**, $\vec{B}(t)$, of our vector . The way to proceed is to **decompose the initial vector**, $\vec{B}_0$, into a piece $\vec{B}_{0\parallel}$ that is **parallel to $\vec{\omega}$** and a piece $\vec{B}_{0\perp}$ that is **perpendicular to $\vec{\omega}$**. A third ingredient is also needed: where will the vector be **after a quarter-cycle** of rotational motion? Call that vector $\vec{B}_1$. What we need is its component perpendicular to $\vec{\omega}$, which we call $\vec{B}_{1\perp}$. As you can see from the figure, $\vec{B}_{1\perp}$ is just a 90° rotation of $\vec{B}_{0\perp}$ around $\vec{\omega}$.

(a) The three key ingredients of our solution are $\vec{B}_{0\parallel}$, $\vec{B}_{0\perp}$, and $\vec{B}_{1\perp}$. Using pure geometry, match them up with these three expressions (i.e. which is which?) ➔ $\hat{\omega}\times\vec{B}_0$, $\vec{B}_0 - (\vec{B}_0\cdot\hat{\omega})\hat{\omega}$, $(\vec{B}_0\cdot\hat{\omega})\hat{\omega}$

(b) Since $\vec{\omega}$ is constant and we know the t=0 starting-value $\vec{B}_0$ of our vector, we can immediately write down the general solution for $\vec{B}(t)$ using cosines and sines: $\vec{B}(t) = \vec{a} + \vec{b}\cos(\omega t) + \vec{c}\sin(\omega t)$. Your task is to figure out what $\vec{a}, \vec{b}, \vec{c}$ are. Hint: each is one of our three "ingredients" $\vec{B}_{0\parallel}$, $\vec{B}_{0\perp}$, $\vec{B}_{1\perp}$.

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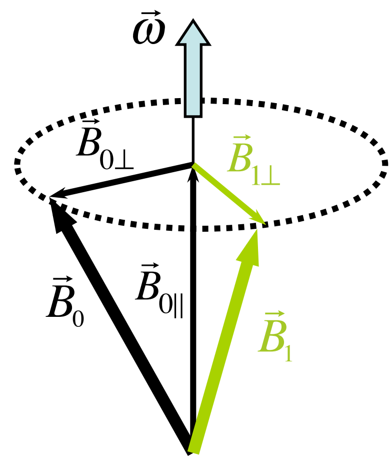
[1] (a),(b) checks coming up (d) checked by (e,f) $\vec{v}(t) = 2\omega\left[-\hat{x}\sin(\omega t) - \hat{z}\cos(\omega t)\right]$

How did you do? The solution is *more checkpoints* [2]



$$\vec{B}(t) = \vec{B}_{0\|} + \vec{B}_{0\perp}\cos(\omega t) + \vec{B}_{1\perp}\sin(\omega t) \text{ where}$$

$$\vec{B}_{0\|} = (\vec{B}_0 \cdot \hat{\omega})\hat{\omega}\ , \qquad \vec{B}_{0\perp} = \vec{B}_0 - \vec{B}_{0\|}\ , \qquad \vec{B}_{1\perp} = \hat{\omega} \times \vec{B}_{0\perp} = \hat{\omega} \times \vec{B}_0\ .$$

This comes entirely from geometry. It is much easier to sketch the figure and recall the general procedure than to try to remember these formulas!

(c) An example is urgently needed! Consider a rigid flat right-triangle with perpendicular sides of length 2 and 1. At time t=0 the triangle is in the xy-plane and its vertices are at these locations: (x,y) = (0,0), (2,0), (0,1).
A blue dot is painted on the vertex at (2,0) and we label this vertex "point B".
It goes without saying that you must draw this thing before going any further!

(d) Suppose the triangle is rotating around the y-axis with angular velocity $\vec{\omega} = \omega\hat{y}$. Write down the complete trajectory of the point B (the blue dot) in the form $\vec{r}(t) = x(t)\hat{x} + y(t)\hat{y} + z(t)\hat{z}$.

(e) Did we actually solve the differential equation $\dot{\vec{r}}(t) = \vec{\omega} \times \vec{r}(t)$? Calculate the left-hand and right-hand sides of this expression – i.e. calculate the blue dot's velocity $\vec{v}(t)$ in two ways – to show that we did! (It is not hard to prove that our geometric solution at the top of the page does solve $\dot{\vec{r}}(t) = \vec{\omega} \times \vec{r}(t)$. Try it if you like!)

(f) If all you care about is the velocity $\vec{v}(t)$ of the blue dot, there is a faster way: apply our general solution to the vector $\vec{v}$ instead of the vector $\vec{r}$! $\rightarrow$ $\vec{v}(t) = \vec{v}_{0\|} + \vec{v}_{0\perp}\cos(\omega t) + \vec{v}_{1\perp}\sin(\omega t)$. Note: you must find $\vec{v}_0$ first.

(g) Now try this: the triangle is rotating around the axis that runs through the origin and has direction $\vec{\omega} \parallel (\hat{x} + 2\hat{y})$. Calculate $\vec{v}(t)$ of the blue dot in this case. If you want more practice, also calculate $\vec{r}(t)$.

Problem 2 : Small Oscillations of a Spinning Book *Checkpoints* [3]

We used Euler's equations today to analyze the rotational stability of a book, and we checked it with a demo. Your turn to analyze the demo! A book of uniform density and dimensions (a=30 cm) × (b=20 cm) × (c=3 cm) is held shut with a rubber band. You throw the book into the air spinning at 180 rpm (revolutions per minute) about an axis that is *very close* to the book's shortest symmetry axis (i.e. the axis parallel to the shortest dimension of the book). What is the frequency of small oscillations of the book's axis of rotation in the book's body-frame? Give your answer in rpm, so you don't have to convert this awkward unit into anything else.

☞ **Euler's Equations** : $\tau_1 = I_1\dot{\omega}_1 + (I_3 - I_2)\omega_2\omega_3$ is all you need ; just cycle the indices to get the other two!

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[2] Q1 (g) $\vec{v}(t) = -\frac{4}{\sqrt{5}}\omega\cos(\omega t)\hat{z} + \frac{4}{5}\omega\sin(\omega t)(-2\hat{x} + \hat{y})$, $\vec{r}(t) = \hat{x}\left[\frac{2}{\sqrt{5}} + \frac{8}{5}\cos(\omega t)\right] + \hat{y}\left[\frac{4}{\sqrt{5}} - \frac{4}{5}\cos(\omega t)\right] + \hat{z}\frac{4}{\sqrt{5}}\sin(\omega t)$

[3] Q2 $\Omega^2 = \omega_1^2\left(\frac{a^2 - c^2}{a^2 + c^2}\right)\left(\frac{b^2 - c^2}{b^2 + c^2}\right)$ $\rightarrow$ $\Omega = \omega_1$ (0.968) = 174 rpm