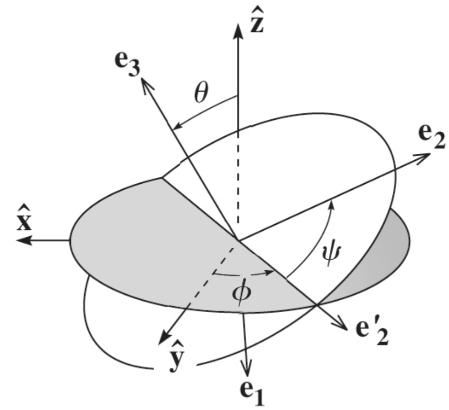
Phys 326 Discussion 11 – Euler Angles



The Euler angles (ϕ,θ,ψ) provide an excellent way of analyzing the general rotation of a rigid body because they can be readily interpreted. Here are the main formulae and the key figure :

$\vec{\omega} = \dot{\phi}\,\hat{z} + \dot{\theta}\,\hat{e}_2' + \dot{\psi}\,\hat{e}_3$ directly from figure

$$\boxed{\begin{aligned}\vec{\omega} &= -\dot{\phi}\sin\theta\,\hat{e}_1' + \dot{\theta}\,\hat{e}_2' + (\dot{\psi} + \dot{\phi}\cos\theta)\,\hat{e}_3 \\ &= \qquad \vec{\omega}_{12} \qquad + \qquad \vec{\omega}_3\end{aligned}}$$

decomposition into ⊥ components using $\hat{z} = \cos\theta\,\hat{e}_3 - \sin\theta\,\hat{e}_1'$

To obtain the angular momentum and kinetic energy (so we can do some problems!), we as usual like to work with body-frame components, since the inertia tensor is diagonal in the body frame. That last decomposition of ω is perfect for this purpose! For an axisymmetric body (i.e. one with $I_1 = I_2$) we write

$$\vec{L} = I_1\vec{\omega}_{12} + I_3\vec{\omega}_3 \quad \text{and} \quad 2T = \vec{\omega}^T\mathbf{I}\vec{\omega} = I_1\omega_{12}^2 + I_3\omega_3^2$$

as usual, then substitute in the Euler-angle expressions for $\vec{\omega}_{12}$ and $\vec{\omega}_3$ from above.

Problem 1 : Free Symmetric Top with Euler Angles *Checkpoints* [1]

Let's use the Euler angles to analyze a system we now know well: the free symmetric top (e.g. a football or the Earth spinning and wobbling). The entire problem can be solved using the fact that, since the object is free (no torques), the vector $\vec{L}$ is constant. Further, let us make $\vec{L}$ point in the z-direction: $\vec{L} = L\hat{z}$. The reason for this choice is simply that the Euler-angle solution will be easier to understand if our constant vector $\vec{L}$ – our "anchor" in the lab frame – points along the "special" axis, $\hat{z}$, of the Euler system.

(a) First, do you understand the geometric relation $\hat{z} = \cos\theta\,\hat{e}_3 - \sin\theta\,\hat{e}_1'$ that was used to get the key $\vec{\omega}$ formula in the box? It comes directly from the picture. These angles do take some getting used to, so stare at the figure and make absolutely sure you can see the relation is true. The relation is entirely equivalent to these three:

$$\hat{z}\cdot\hat{e}_3 = \cos\theta\ , \quad \hat{z}\cdot\hat{e}_2' = 0\ , \quad \hat{z}\cdot\hat{e}_1' = -\sin\theta$$

Make sure that statement makes sense too!

(b) OK! Now write down $\vec{L} = I_1\vec{\omega}_{12} + I_3\vec{\omega}_3 = L\hat{z}$ for constant L using the $\vec{\omega}$ formula in the box and the $\hat{z}$ relation you just figured out. The $\vec{\omega}$ formula in the box has the huge advantage that its three components are orthogonal to each other ... so match components on each side of your equation and to get three expressions.

(c) What is ω_3 in terms of the Euler angles?

(d) Compare the three formulae you just obtained to the free-symmetric top quantities

$$\vec{\Omega} = \vec{L}/I_1 \quad \text{and} \quad \vec{\Omega}^* = \left[(I_3/I_1) - 1\right]\omega_3\hat{e}_3\ .$$

Express these two precession frequencies in terms of the Euler angles and see if it makes sense!

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[1] (b) $\hat{e}_1'$ components give $\dot{\phi} = L/I_1$ (or the trivial solution $\theta = 0$) … $\hat{e}_2'$ components give $\dot{\theta} = 0$, i.e. θ = constant … $\hat{e}_3$ components give $\dot{\psi} + \dot{\phi}\cos\theta = L\cos\theta / I_3$ (c) $\omega_3 = \dot{\psi} + \dot{\phi}\cos\theta$ (d) $\Omega = \dot{\phi}$, $\Omega^* = -\dot{\psi}$

Problem 2 : Spinning Top on a Table *Hints & Checkpoints* [2]

Euler's angles provide us with perfect generalized coordinates for a rigid body, and we can use them to build a Lagrangian. Hurray! Let's analyze the motion of an axisymmetric top spinning on a table in uniform gravity g. Put the origin at the location of the fixed pivot point on the table, define $\hat{e}_3$ to point along the axis of symmetry as usual, and let R be the distance from the pivot to the CM of the top. We also know the moments of inertia I_3 and $I_1 = I_2$ of the top and its total mass M. Our task is to figure out its general motion.

(a) We need the Lagrangian $\mathcal{L}(\phi,\theta,\psi,\dot{\phi},\dot{\theta},\dot{\psi}) = T - U$ for this system. Write it down! ☺

(b) Your Lagrangian should have two cyclic coordinates: ψ and ϕ. These will give constants of motion, which is great! First, find the constant generalized momentum $p_\psi \equiv \partial\mathcal{L}/\partial\dot{\psi}$ associated with the ψ coordinate.

(c) Generalized momenta associated with angles are usually some form of angular momentum. p_ψ is no exception. Which component of angular momentum is it? You have a lot of choices: L_x, L_y, L_z, L_1, L_2, L_3, $L_{1'}$, $L_{2'}$, … but it *is* one of these. ☺

(d) Now find the constant generalized momentum $p_\phi \equiv \partial\mathcal{L}/\partial\dot{\phi}$ associated with the ϕ coordinate. Once you have your expression, simplify it a bit using the constant $\omega_3 = \dot{\psi} + \dot{\phi}\cos\theta$ you obtained from (b),(c).

(e) What is the significance of this p_ϕ? Again it is one of the components of $\vec{L}$, but it is not so easy to pick out. Take a guess: p_ϕ is associated with the motion $\dot{\phi}$ around the z-axis ... so it is probably L_z. Start with the general expression $\vec{L} = I_1\left(-\dot{\phi}\sin\theta\,\hat{e}'_1 + \dot{\theta}\,\hat{e}'_2\right) + I_3\omega_3\,\hat{e}_3$, calculate $L_z = \vec{L}\cdot\hat{z}$, and show that indeed $p_\phi = L_z$.

(f) Our third and final EOM is the one associated with θ. Write it down.

(g) This 2nd order equation is not trivial to solve. Let's take a special case: θ = constant. This is a guess … try it and see if it works. What conclusion do you reach for the time-dependence of $\dot{\phi}$ and $\dot{\psi}$?

(h) Let's introduce the label Ω for the constant $\dot{\phi}$. Knowing only that θ, $\dot{\phi}$, and $\dot{\psi}$ are constant, describe the motion of the top and the significance of Ω in that motion.

(i) Now consider the further special case where ω_3 is very big, i.e. far larger than any other quantity with the same dimensions). This is the usual situation when you spin a top : the rotation $\vec{\omega}$ you give it is as close to the $\hat{e}_3$ symmetry axis as possible, so $\omega \approx \omega_3 >> \omega_1, \omega_2$. Using your EOM from part (g), solve for the constant procession frequency $\Omega \equiv \dot{\phi}$. You will obtain a quadratic equation with two solutions: one large, and one small. Use the $\omega_3 >>$ everything approximation to find these two solutions.

(j) The big precession frequency doesn't involve gravity at all! Not surprisingly, it is one of the precession frequencies of the free symmetric top. Which one is it?

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[2] (a) $\mathcal{L} = T - U$ with $2T = I_1\left(\dot{\phi}^2\sin^2\theta + \dot{\theta}^2\right) + I_3\left(\dot{\psi} + \dot{\phi}\cos\theta\right)^2$ and $U = MgR\cos\theta$ (b) $p_\psi = I_3\left(\dot{\psi} + \dot{\phi}\cos\theta\right)$

(c) using Q1(c), $p_\psi = I_3\omega_3 = L_3$ = constant … so $\omega_3 = \dot{\psi} + \dot{\phi}\cos\theta$ is constant too. (d) $p_\phi = I_1\dot{\phi}\sin^2\theta + I_3\omega_3\cos\theta$

(e) hint: use the dot-products from Q1(a) (f) $I_1\ddot{\theta} = \sin\theta\left(\dot{\phi}^2 I_1\cos\theta - \dot{\phi} I_3\omega_3 + MgR\right)$ (g) $\dot{\phi}, \dot{\psi}$ both constant (h) top spins around $\hat{e}_3$, and $\hat{e}_3$ precesses around $\hat{z}$ at rate Ω (i) $\Omega_{\text{BIG}} \approx I_3\omega_3/(I_1\cos\theta)$ & $\Omega_{\text{SMALL}} \approx MgR/(I_3\omega_3)$

(j) $\Omega_{\text{BIG}} \approx (L_3/\cos\theta)(1/I_1) = L/I_1 = \Omega$ = FST precession frequency of $\hat{e}_3$ around $\hat{z}$ relative to the lab frame