Phys 326 Discussion 14 – Waves

We derived the differential equation for waves on a transverse string, but the form of the equation appears in countless places in physics. For a generic "disturbance" u, i.e. some quantity u that represents a local deviation from equilibrium at time t and position x (1D) or $\vec{r}$ (3D), the **wave equation** is

1D $u(x,t)$: $\dfrac{\partial^2 u}{\partial t^2} = c^2 \dfrac{\partial^2 u}{\partial x^2}$ 3D $u(\vec{r},t)$: $\dfrac{\partial^2 u}{\partial t^2} = c^2 \nabla^2 u$

The general solution of the 1D wave equation can be decomposed in a couple of ways. All waves can be decomposed into **forward- & backward-going** "waves" = disturbances of any shape:

- $u(x,t) = f(x-ct) + g(x+ct)$ The arguments $(x \pm ct)$ are called the **phase** of the shapes f and g.

Waves within an object of finite length x=0 to x=L where both ends are "clamped" to u = 0 can be written as a sum over **normal modes** of frequency ω_m:

- $u(x,t) = \sum_{m=1}^{\infty} \left(\sin k_m x\right)\left(A_m \cos \omega_m t + B_m \sin \omega_m t\right)$ where $k_m = \dfrac{\omega_m}{c} = \dfrac{m\pi}{L}$

Waves in an infinite medium without such boundary conditions can also be decomposed into such Fourier terms … you just need an integral over frequency rather than a sum, and $\cos(kx) = \cos(\omega x / c)$ terms may also appear.

Problem 1 : Reflection at a Boundary *Checkpoints* [1]

A semi-infinite string runs from x=0 to x=∞. The string's transverse displacement from the x axis is given by $u(x,t)$ and this u satisfies the wave equation with speed c. The string is clamped at the x=0 end: $u(0, t) = 0$ at all times t.

(a) Let's define a shape for the disturbance we will introduce:

$$h(\phi) = \max\left(1 - \frac{|\phi|}{a}, 0\right)$$ where a is some constant with units of length.

Sketch the function $h(\phi)$.

(b) At some negative time $t < 0$, we flick the string so as to produce the waveform $u(x,t) = h(x+ct)$.
Describe the behaviour of the string at some large negative time $t_0 \ll -a/c$:

- What does the string look like at t_0?
  (You might find it helpful to introduce the symbol x_0 for the point of maximum displacement.)
- How is the string's profile going to *change* as time moves to the next moment $t_0 + \Delta t$?

(c) Is the wave $u(x,t) = h(x+ct)$ actually a *solution* to this problem? It certainly solves the wave equation but does it satisfy the boundary condition "$u(0, t) = 0$ at all times t"? Does it satisfy the boundary condition at *any* times t?

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[1] **(a)** A triangle that extends from $\phi = -a$ to $+a$ **(b)** Wave peak is at $x_0 = -c\,t_0 \gg a$; peak & shape will move to the left at speed c. **(c)** only works at times $|\,ct\,| > a$, i.e. not near times $t = 0$ **(d)** $g(x-ct) = -h(x+ct)$ **(e)** no … for times $ct_0 << -a$, peak position of second term is at $x_0 = ct_0 \ll -a$, which is not even on the string. **(f)** enjoy!

(d) Fortunately there is an easy way to fix this: we can add to our solution a second term $g(x-ct)$.
Figure out what g has to be to satisfy the condition " $u(0,t)=0$ at all times t".

(e) Has this new term affected the original disturbance we applied to the string at large negative time?
To figure it out, figure out the peak position x_0 of the second term $g(x-ct)$ at a large negative time $t_0 \ll -a/c$.

(f) Now sketch the entire wave $u(x,t)=h(x+ct)+g(x-ct)$ at five different times:

$$t=\left\{-\frac{a}{c},\ -\frac{a}{2c},\ 0,\ +\frac{a}{2c},\ +\frac{a}{c}\right\}$$

<u>HINT</u>: For this exercise, you will find it helpful to draw a "ghost" extension of the string into the region $x<0$. Draw the x-axis from –3a to +3a, draw the waveforms h and g at each of the given times, add them together, then remember that there is no actual string in the "ghost" region to the left of the origin so you can ignore it.

Message 1: You have discovered a general principle: **waves reflect from boundaries between materials**. By "reflect" we mean "bounce back", and by "material boundary" we mean a boundary where the **characteristic speed of the waves changes**. Recall that for a transversely-disturbed taut string, $c^2 = T/\varrho$ where T is the string tension at equilibrium and ϱ is the mass density of the string; the characteristic wave speed c is thus an innate property of the string. The material boundary in this problem is the most extreme of all: c^2 goes from T/ϱ in the region $x>0$ to *undefined* in the region $x<0$ where there is no string!

Message 2: You have also discovered that **waves flip upside down when they reflect** from a boundary. You have likely already seen this behaviour in at least one other context, e.g. electromagnetic waves travelling from air into glass.

Problem 2 : Spherical Wave *Checkpoints* [2]

The 3D wave equation can be solved in spherical coordinates to produce a particular type of wave that propagates only in the inward and outward <u>radial</u> directions → a **spherical wave**. This wave is of the form $u(r,t)$, i.e. the disturbance u is spherically symmetric and so does not depend on the angles θ or ϕ. Writing down the 3D wave equation and substituting in the r-dependent terms of the Laplacian from your 3Dmath formula sheet, we get:

$$\frac{1}{c^2}\frac{\partial^2 u}{\partial t^2}=\nabla^2 u=\frac{1}{r}\frac{\partial^2}{\partial r^2}(ru) \quad \text{for} \quad u(r,t)$$

Find the general solution of this wave equation that involves oscillations (rather than exponentials). Step-by-step hints are in the footnote. One simple hint straight away: do <u>not</u> expand the $\partial^2(ru)/\partial r^2$ derivative, leave the combination ru together all the way to the final step. Also, make use of the convenient variables ω and k, which are always related by $c=\omega/k$.

[2] Hint 0 : use the PDE sledge-hammer technique "separation of variables" … Hint 1: which means guess this form : $u(r,t) = R(r)\,T(t)$ … Hint 2: after plugging in $u = RT$, divide both sides of the equation by RT … Hint 3: both sides should now be very obviously equal to a constant ... Hint 4: make that constant negative so that so you get oscillating solutions …
Solution: $u(r,t)=\frac{1}{r}\left(A\cos kr+B\sin kr\right)\left(C\cos\omega t+D\sin\omega t\right)$