Phys 326 Discussion 15 – Stress and Strain in a Solid

The **stress tensor Σ** (or σ_{ij} when index notation is in use) describes the surface forces per unit area on a differential element of area $d\vec{A}$ within a material. $d\vec{A}$ always points outward from the element experiencing the force. All this is summarized by the somewhat wordy equation

$$\vec{F}^{\text{surface}}_{\substack{\text{acting on}\\ \text{IN side}}} = \mathbf{\Sigma}\, d\vec{A}_{\text{OUT}} \qquad \ldots \text{ or in index notation:} \quad F_i^{\text{surface}} = \sigma_{ij} dA_j$$

The **strain tensor E** (or ε_{ij} in index notation) describes the fractional deformation of a material under stress. When stress is applied to a tiny element of material at location $\vec{r}$, it moves from its original position $\vec{r}$ to a slightly shifted position $\vec{r}' = \vec{r} + \vec{u}(\vec{r},t)$. The function $\vec{u}(\vec{r},t)$ is called the **displacement field** of the material.

no strain: element at $\vec{r}$ $\quad\rightarrow\quad$ under strain: element moves to $\vec{r}' = \vec{r} + \vec{u}(\vec{r},t)$

We assuming all displacements are small (as they generally are in a solid). The strain tensor is directly related to the displacement field : it is a symmetrized version of the **derivative matrix D** :

$$\mathbf{D}_{ij} \equiv \frac{\partial u_i}{\partial r_j} = \begin{pmatrix} \partial_x u_x & \partial_y u_x & \partial_z u_x \\ . & . & . \\ . & . & . \end{pmatrix} \quad \rightarrow \quad \mathbf{E} \equiv \frac{\mathbf{D} + \mathbf{D}^T}{2} = \begin{pmatrix} \partial_x u_x & \frac{1}{2}\left(\partial_y u_x + \partial_x u_y\right) & . \\ \frac{1}{2}\left(\partial_x u_y + \partial_y u_x\right) & \partial_y u_y & . \\ . & . & . \end{pmatrix}$$

(Here I'm using the familiar shorthand $\partial_j u_i \equiv \partial u_i / \partial r_j$.) As we learned, "stress / strain = elastic modulus", for various types of stress, strain and modulus (BM=Bulk, YM=Young's, SM=Shear Modulus). The formal relation between the stress and strain tensors is the **generalized Hooke's Law**. Here it is in "$F = k\Delta x$" and "$\Delta x = F/k$" forms, where I am making an analogy between the familiar F, k, Δx for a spring and our new **Σ**, M, **E** quantities.

- $\mathbf{\Sigma} = (\alpha - \beta) e\, \mathbf{1} + \beta\, \mathbf{E}$
- $\mathbf{E} = \frac{1}{3\alpha\beta}\left[3\alpha\, \mathbf{\Sigma} - (\alpha - \beta)(\text{tr}\, \mathbf{\Sigma})\, \mathbf{1}\right]$

where $\alpha = 3\text{BM}$, $\beta = 2\text{SM}$, $e = \frac{1}{3}\text{tr}\, \mathbf{E}$ and $\text{YM} = \frac{3\alpha\beta}{2\alpha + \beta} = \frac{9\text{BM} \cdot \text{SM}}{3\text{BM} + \text{SM}}$

Problem 1 : An Example Stress Tensor *Checkpoints* [1]

It is found that the stress tensor in a continuous medium has the form given at right : (Ignore the units ☺.) Find the surface force $\vec{F}^{\text{surf}}$ on a small area dA of the surface $x^2 + y^2 + 2z^2 = 4$ at the point $(x,y,z) = (1,1,1)$.

$$\mathbf{\Sigma} = \begin{pmatrix} xz & z^2 & 0 \\ z^2 & 0 & -y \\ 0 & -y & 0 \end{pmatrix}$$

Problem 2 : Longitudinal Waves on a String *Checkpoints* [2]

A taut, massive string can support longitudinal waves as well as transverse ones. Suppose we have a string of cross-sectional area A, made of a material with Young's Modulus YM, and lying along the x axis. The displacement field for this string is $\vec{u}(\vec{r},t) = u_x(x,t)\hat{x}$ since this problem is entirely 1-dimensional.

(a) Consider a very short piece of string of length l. We defined Young's Modulus as follows:

---

[1] $\vec{F} = dA\left(2\hat{x} - \hat{y} - \hat{z}\right)/\sqrt{6}$

[2] (b) $\frac{\partial^2 u_x}{\partial t^2} = \frac{YM}{\rho}\frac{\partial^2 u_x}{\partial x^2}$ (c) $c = \sqrt{YM/\rho}$

$$YM = \frac{\text{tension-stress}}{\text{tension-strain}} \qquad \text{where tension-stress} = \frac{T}{A_\perp} \text{ and tension-strain} = \frac{\Delta l}{l}$$

Using primes to denote conditions under strain, as in $\vec{r} \;\rightarrow\; \vec{r}' = \vec{r} + \vec{u}(\vec{r},t)$, the tension-strain can be written $\Delta l / l = (l' - l)/l$. Considering what happens to the ends of the string under strain, show that the tension is related to Young's Modulus by $T = A(YM)\partial u / \partial x$.

(b) Now consider the forces on the ends of the little piece of string. Using $F = ma$, find the equation of motion for the longitudinal disturbance $u_x(x,t)$. Hint: to help you find the differential equation you seek, change notation a bit → change the string segment's length from l to dx and put its center at some position x.

(c) What is the speed of longitudinal wave propagation along this string?

Problem 3 : Young's Modulus Relation *Checkpoints* [3]

We didn't impose any conditions on the cross-sectional area A of the string in problem 2 so the string can be a solid block. Suppose the tension in the block of material in the x-direction only (as for the string) and constant : $\vec{T}(\vec{r}) = T\,\hat{x}$.

(a) Write down the stress tensor $\mathbf{\Sigma} = \sigma_{ij}$ for this block under constant tension. Then use the "$\Delta x = F/k$" form of generalized Hooke's Law (the second bullet on the previous page) to write down the strain tensor $\mathbf{E} = \varepsilon_{ij}$ in terms of the material constants α and β and the one non-zero stress-tensor element σ_{ij} you obtained.

(b) Using the tension-stress & tension-strain analysis you already performed in problem 2, convince yourself that Young's Modulus is YM = one-element-of-σ / one-element-of-ε. (which elements?)

(c) Combine these results to derive the expression on the previous page for YM in terms of α and β.

---

[3] (a) $\mathbf{E} = \frac{\sigma_{11}}{3\alpha\beta}\begin{pmatrix} 2\alpha+\beta & 0 & 0 \\ 0 & \beta-\alpha & 0 \\ 0 & 0 & \beta-\alpha \end{pmatrix}$ (b) $\text{YM} = \frac{\sigma_{11}}{\varepsilon_{11}}$