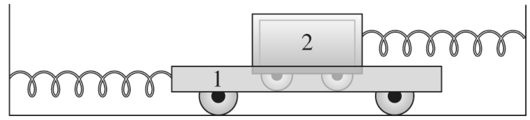
Physics 326 – Homework #3 **hard deadline extended to WEDNESDAY, 10:00 am**

Problem 0 : Not for points on this homework, but please do not forget to work through Discussion 3 Problem 2 (which very few got to), which covers degenerate modes.

Problem 1 : Drag Coupling



In homework 2, we solved coupled oscillators with a damping force involved. We used normal coordinates to solve it: that decoupled the equations of motion, reducing the problem to the familiar damped oscillator with 1 degree of freedom from PHYS 325 / MATH 285. Let's try a different method: this time we will use matrix notation to solve a damped oscillator in all n of its dimensions at once.

The two carts in the figure have equal masses m. They are joined by identical but separate springs of force constant k to separate walls. Cart 2 rides in cart 1 as shows, and cart 1 is filled with molasses, whose viscous drag supplies the coupling between the two carts. The drag force has magnitude $\beta m v$ where v is the relative velocity of the two carts.

(a) Write down the equations of motion of the two carts using as coordinates x_1 and x_2, the displacements of the carts to the right of their equilibrium positions. Show that the EOM can be written in matrix form as $\mathbf{1}\ddot{\vec{x}} + \beta\mathbf{D}\dot{\vec{x}} + \omega_0^2\mathbf{1}\vec{x} = 0$, where $\vec{x}$ is the column vector made up of x_1 and x_2, $\omega_0 \equiv \sqrt{k/m}$, $\mathbf{1}$ is the unit matrix, and $\mathbf{D}$ is a certain 2×2 square matrix for you to determine.

(b) The next step is to "guess the solution form". Let's try normal mode form, but with a slight variation. **Normal mode form** means a solution where all the coordinates are oscillating at the same frequency and the same phase. This system has damping, however, so its oscillations will decay with time. That suggests a solution form $\vec{x}(t) = \vec{a}\,e^{\tilde{\omega}t}$ where we hypothesize a **common frequency** $\tilde{\omega}$ that is **complex** instead of the usual $i\omega$. Assuming that the drag force is weak ($\beta < \omega_0$), show that you do get two solutions of this form with $\tilde{\omega} = i\omega_0$ or $\tilde{\omega} = -\beta + i\sqrt{\omega_0^{\,2} - \beta^2}$. HINT: The determinant factorizes, stare at it until you see it!

(c) Describe the corresponding motions. Explain why one of these modes is damped but the other is not.

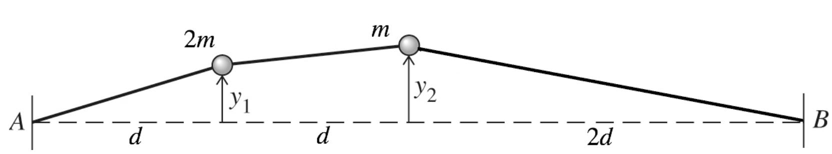
Problem 2 : 3 Beads and Springs on a Ring *Qual Problem*

Consider a frictionless rigid horizontal hoop of radius R. Onto this hoop we thread three beads with masses $2m$, m, and m; between the beads we thread three identical springs on the hoop, each with force constant k.

(a) Solve for the three normal frequencies.

(b) Find the three normal modes, describe them with sketches, and express them in normalized form, i.e. so that their amplitude vectors obey the orthonormality relation $\langle \hat{a}_m | \hat{a}_n \rangle = \hat{a}_m^T \mathbf{M}\,\hat{a}_n = \delta_{mn}$. Take $R = 1$ for simplicity.

Problem 3 : Transverse Modes



Two particles, of masses $2m$ and m, are secured to a light string of total length $4d$ that is stretched to tension T_0 between two fixed supports. As shown in the figure, the masses are not evenly spaced along the string. The masses undergo small transverse oscillations, where their transverse displacements from equilibrium, y_1 and y_2, are kept to very small values compared with the length-scale d of the string.

(a) Find the normal frequencies of transverse oscillation for this system. You will find it useful throughout this problem to define the constant $\alpha \equiv T_0 / (dm)$ → using it will greatly simplify your expressions!

(b) Write down the general solution for $y_1(t)$ and $y_2(t)$.

(c) Is the general motion you calculated in (b) periodic? Explain why or why not, and if it is, give the period of the general motion.

(d) Normalize the eigenvectors for the fast and slow modes to obtain an orthonormal basis $\{\hat{a}_F, \hat{a}_S\}$.

(e) Find the normal coordinates ξ_F and ξ_S in terms of the generalized coordinates y_1 and y_2, and determine the matrices $\mathbf{R}$ and $\mathbf{R}^{-1}$ that relate them via $\vec{\xi} = \mathbf{R}\,\vec{y}$ and $\vec{y} = \mathbf{R}^{-1}\vec{\xi}$.

(f) Explicitly transform the mass matrix $\mathbf{M}$ and spring matrix $\mathbf{K}$ to ξ–space (i.e., calculate $\mathbf{M}^\xi$ and $\mathbf{K}^\xi$) using the matrix transformation formula $\mathbf{M}^\xi = \left(\mathbf{R}^{-1}\right)^T \mathbf{M}\left(\mathbf{R}^{-1}\right)$ derived in class, and verify that they are diagonal.

(You do not have to rederive the diagonal form again, just this once. ☺)

Problem 4 : Driven 3m2s System *Qual Problem*

Three identical blocks of mass $m = 1$ are placed in a line on a frictionless horizontal table and connected by identical springs of spring-constant $k = 1$. With the $+x$ direction pointing to the right, we number the blocks as 1,2,3 from left to right, and define x_1, x_2, and x_3 to be their x-positions relative to equilibrium. The blocks are initially at rest at $x_1 = x_2 = x_3 = 0$. At time $t = 0$, an external driving force $\vec{F} = f\cos(\omega t)\,\hat{x}$ is applied to block 1. Calculate $x_3(t)$ = the motion of block 3 for times $t \geq 0$. Tactics: You have the same basic choice to make in this driven-oscillator problem as with a damped-oscillator problem → *do you switch to normal coordinates or not*? It is a tradeoff. Here is a little summary of what will happen if you use ξ or not:

Step	Not using ξ	Using ξ
(1) Find homogeneous solution	*usual procedure, same in both methods*	
(2) Find particular solution	easy	transformation algebra : go to ξ-space
(3) Apply initial conditions	horrible algebra	easy
(4) → final solution for $x_3(t)$	trivial	transformation algebra : return to x-space

Of course the best thing is to try *both* methods and see which you prefer. :-)

Problem 5 : 4-Atom Ring Molecule *Qual Problem*

To study the vibrational spectrum of a ring molecule like benzene, one can reasonably approximate the molecule's atoms / sub-molecules as beads placed on a ring with springs between them. Let's try a 4-element ring molecule: consider four identical beads of mass m placed on a ring with springs of equal strength k running along the ring between the beads. Using as generalized coordinates the positions x_1, x_2, x_3, x_4 of the 4 beads measured along the ring relative to equilibrium, determine the four normal modes of the system. Provide a small sketch of each mode so you can visualize it, and make sure your four modes are orthogonal to each other.

Hint: the 4x4 matrix $\mathbf{M}\omega^2 - \mathbf{K}$ can be hugely simplified by introducing a variable $\alpha \equiv (m\omega^2/k - 2)$.