Physics 326 – Homework #5 due Friday, 1 pm

Problem 1 : The Foci of an Ellipse ➔ We Will Need This Soon !

The **focal points** of an ellipse are defined in various ways. Here is a really nice definition: if you place two pins in a piece of cardboard and tie the ends of a fixed-length string to the pins, then stretch the string by pushing it outward with a pen, the shape you draw by moving the pen around to all points you can reach without breaking the string will be an ellipse. The pins are the two focal points and the spacing between them is usually labelled $2c$ (i.e. each pin is a distance c from the origin at the center of the ellipse), the length of the string is $2a$ = the length of the major axis. Your tasks:

(a) Show that if you set up a Cartesian coordinate system with the midpoint between the pins as the origin, the curve your pen traces out is indeed the most familiar equation for an ellipse: $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, with $c^2=a^2-b^2$.

(b) Show that if you set up a polar coordinate system with the focal point on the $+x$ side as the origin, then the curve your pen traces out can be written as follows: $\frac{1}{r}=\frac{a}{a^2-c^2}\left(1+\frac{c}{a}\cos\theta\right)$. We will encounter this form very soon and you must recognize it! FYI: If we define the **eccentricity** $\boldsymbol{e}$ via $c=ae$, we get the more common form $\frac{1}{r}=\frac{a}{b^2}(1+e\cos\theta)$ for an ellipse.

(c) In (b), I explicitly used θ as the angular variable instead of the usual ϕ to stress that this form only applies if you place the origin at one of the focal points, NOT at the center of the ellipse! To stress this unfamiliar arrangement, write down the equation for the ellipse using polar coordinates (r,ϕ) that are centered on the origin — just transform your result from part (a) — and compare your expression to the focal-point centered form $(1/r)=(a/b^2)(1+e\cos\theta)$ ➔ they are *not* the same!

Problem 2 : Adding an External Force

Although the main topic of this section is the motion of two particles subject to *no* external forces, the key strategy of decomposing the Lagrangian into two independent pieces, $\mathcal{L}=\mathcal{L}_{CM}+\mathcal{L}_{REL}$, extends easily to more general situations. To illustrate this, consider the following: two masses m_1 and m_2 move in a uniform gravitation field $\vec{g}=-g\hat{z}$ and interact via a potential energy $U(r)$.

(a) Show that the Lagrangian $\mathcal{L}(\vec{r}_1,\vec{r}_2,\dot{\vec{r}}_1,\dot{\vec{r}}_2)$ for the two masses can be rewritten so that it decomposes in the form $\mathcal{L}(\vec{R},\vec{r},\dot{\vec{R}},\dot{\vec{r}})=\mathcal{L}_{CM}(\vec{R},\dot{\vec{R}})+\mathcal{L}_{REL}(\vec{r},\dot{\vec{r}})$ where $\vec{R}$ is the CM position and $\vec{r}\equiv\vec{r}_1-\vec{r}_2$ is the relative position of the two masses. You may use Cartesian components (X,Y,Z) for $\vec{R}$, or you can cast your expression in coordinate-free form by using the vector $\vec{g}$ and a suitable dot-product. **Warning**: Be careful not to confuse the symbols $|\dot{\vec{r}}|$ and $\dot{r}$ $\rightarrow$ the first means $|d\vec{r}/dt|$ while the second means $d|\vec{r}|/dt$, and these are not the same thing! Switching from "r-dot" to "v" can help avoid this notational pitfall: $|\dot{\vec{r}}|=v$ while $\dot{r}=v_r$ $\rightarrow$ clearly different.

(b) Write down Lagrange's equations for the three CM coordinates X ,Y, Z and describe the motion of the CM. Write down the Lagrange equations for the relative coordinates and show that the motion for $\vec{r}$ is the same as that of a single particle of mass μ (the reduced mass), with position $\vec{r}$ and potential energy $U(r)$. **Hint**: Pick a coordinate system for the components of $\vec{r}$ (Cartesian or spherical would be best), find the EOM for each coordinate, then show that the 3 EOMs correspond to the expression " $-\vec{\nabla}U(r) = \mu\ddot{\vec{r}}$ " … which is exactly " $\vec{F} = m\vec{a}$ " for a particle of mass μ in a potential $U(r)$.

(c) Two particles of masses m_1 and m_2 are joined by a massless spring of unstretched length L and force constant k. Initially, m_2 is resting on a table and m_1 is held vertically above it at height L. At time $t = 0$, m_1 is tossed vertically upward with velocity v_0. Defining $z = 0$ to be the table and $+z$ to point upward, find the positions $z_1(t)$ and $z_2(t)$ of the two masses at any subsequent time t, before either mass returns to the table, and describe the motion. (Assume v_0 is small enough that m_1 & m_2 don't collide.) **Strategy**: The coordinate switching we do in this problem is conceptually identical to what we did with normal coordinates. The problem & initial conditions are given in <u>z_1, z_2-space</u> ... but <u>z, Z-space</u> is much better for obtaining a solution as it **decouples** the EOMs. Our strategy is thus "switch to z, Z-space … solve and apply initial conditions … switch back to z_1, z_2-space for final answer". This time, no matrices are required. ☺

Problem 3 : Circular Orbits in Different Central Potentials

The behavior of a pair of objects interacting via a central force is very different for different forms of that central force. For example, we will soon prove that <u>gravity</u> (or any other attractive $1/r$ potential) produces closed orbits of elliptical shape, which includes circles as one possibility. Oddly enough, a <u>spring force</u> between the two masses (or any other attractive r^2 potential) also produces ellipses, but this is <u>*not*</u> true for other forms of the central potential. Let's make this completely clear by exploring different forms for $U(r)$.

For all of the following parts, consider a particle of reduced mass μ orbiting in a central force with $U = kr^n$ where $kn > 0$. (Note: n will sometimes be negative in the parts below; in those cases, the $kn > 0$ condition means that k will be negative too.) You can use without proof the formulae we derived in discussion / lecture:

- **Coordinates** : $\vec{r} \equiv \vec{r}_1 - \vec{r}_2$, $M\vec{R} = m_1\vec{r}_1 + m_2\vec{r}_2$ … $\vec{r}_1 = \vec{R} + \frac{m_2}{M}\vec{r}$, $\vec{r}_2 = \vec{R} - \frac{m_1}{M}\vec{r}$
- **L-Equation** : $\dot{\phi} = \frac{L}{\mu r^2}$
- **E-Equation** : $E = \frac{1}{2}\mu\dot{r}^2 + U + \frac{L^2}{2\mu r^2}$
- **Reduced Mass** : $\mu = \frac{m_1 m_2}{M}$
- **Force** Equation : $\mu\ddot{r} = F(r) + F_{\text{cf}}(r)$
- **Effective Potential** : $U^* \equiv U + U_{\text{cf}}$
- **Centrifugal** force & PE : $\vec{F}_{\text{cf}} = \frac{L^2}{\mu r^3}\hat{r}$, $U_{\text{cf}} = \frac{L^2}{2\mu r^2}$

(a) Explain what the condition $kn > 0$ tells us about the force.

(b) Sketch the effective potential energy $U^*(r)$ (written U_{eff} in Taylor's textbook) for these four cases: $n = 2, -1, -2, -3$. Treat the angular momentum L as a known, fixed constant.

(c) Find the radius r_0 at which the particle (with given angular momentum L) can orbit at a fixed radius in a central potential $U = kr^n$. In which of the cases $n = 2, -1, -2, -3$ that you sketched in (b) is such an orbit possible at all? (Rephrased: in which cases can the particle *remain* at some r_0 indefinitely.

(d) Assuming that a fixed-radius = circular orbit *is* possible, for what values of n is this orbit <u>stable</u>? Do your sketches confirm this conclusion?

(e) For the stable case, show that the period of small oscillations about the circular orbit is $\tau_{OSC} = \tau_{ORB} / \sqrt{n+2}$, where τ_{ORB} is the particle's orbital period = the time it takes to complete one circular orbit.

(f) Finally, explore your result from (e) in these three ways:

(f1) For what type of force (gravity, spring-force, …) is τ_{OSC} equal to τ_{ORB} ?

(f2) Argue that if $\sqrt{n+2}$ is a rational number, these perturbed, small-oscillation orbits are closed

(f3) Sketch the orbits for the cases that $n = 2, -1$, and 7 (yes, seven ☺)