

Physics 326 – Homework #6

due FRIDAY, 1 pm

Here is our growing collection of useful formulae concerning the two-body central force problem.

• **Coordinates & Reduced Mass** : $\vec{r}_1 = \vec{R} + \frac{m_2}{M} \vec{r}$, $\vec{r}_2 = \vec{R} - \frac{m_1}{M} \vec{r}$, $\mu = \frac{m_1 m_2}{M}$

• **Centrifugal force & PE** : $\vec{F}_{cf} = \frac{L^2}{\mu r^3} \hat{r}$, $U_{cf} = \frac{L^2}{2\mu r^2}$, effective radial $U^* = U + U_{cf}$

• **Angular EOM** : $\dot{\phi} = \frac{L}{\mu r^2}$ • **Radial EOMs** : $\mu \ddot{r} = F(r) + F_{cf}(r)$, $E = T + U(r) = \frac{1}{2} \mu \dot{r}^2 + U_{cf}(r) + U(r)$

• **Path Equation** : $u(\phi) \equiv \frac{1}{r(\phi)} \rightarrow u'' + u = -\frac{\mu F(1/u)}{L^2 u^2}$ and $u' = -\frac{\mu \dot{r}}{L}$

• **Ellipse / Hyperbola** with (r, ϕ) centered on a focal point : $\frac{1}{r} = \frac{a}{b^2} (1 + e \cos \phi)$, $e = \frac{c}{a} = \frac{\sqrt{a^2 \mp b^2}}{a}$

• **Kepler Orbits** for $F = -\frac{\gamma}{r^2}$: $r(\theta) = \frac{r_0}{1 + e \cos \phi}$ with $r_0 = \frac{L^2}{\mu \gamma} = \frac{b^2}{a}$, $E = \mp \frac{\gamma}{2a} = \frac{\gamma(e^2 - 1)}{2r_0}$, $\tau^2 = \frac{4\pi^2 \mu}{\gamma} a^3$

If you are doing this homework before Tuesday's lecture, we haven't derived that last equation for the period, but do use it!

Problem 1 : Satellite I

The height of a satellite at perigee is 300 km above the earth's surface and it is 3000 km at apogee.

EARTH DATA: For this problem and the ones following, you may need the radius of the Earth; it is $R_{\oplus} = 6.4 \times 10^6$ m. If you need the mass M_{\oplus} of the Earth it will almost certainly be in the combination GM_{\oplus} which is equal to gR_{\oplus}^2 ; use that and the familiar value $g = 9.8$ m/s². Finally, the mass of all satellites can be considered negligible compared to the mass of the Earth.

(a) Find the orbit's eccentricity e .

(b) If we take the orbit to define the xy plane and the major axis in the x direction with the earth at the origin, what is the satellite's height above the earth's surface when it crosses the y axis?

Problem 2 : Satellite II

An earth satellite is observed at perigee to be 250 km above the earth's surface and travelling at about 8500 m/s.

(a) Find the eccentricity of its orbit.

(b) Find its height above the earth's surface at apogee.

Problem 3 : Sputnik

In Yuri Gagarin's first manned space flight in 1961, the perigee and apogee were 181 km and 327 km above the earth's surface.

(a) Find the period of his orbit.

(b) Find his maximum speed in the orbit.

Problem 4 : The Earth Becomes a Star

- (a) Using simple mechanics, find the period of a mass m_1 in a circular orbit of radius r around a fixed mass m_2 . **Note:** “Use simple mechanics” means “don’t use the formulae on the previous page”. Why? Because this situation is not described by those formulae! This situation has one of the actual, physical masses fixed in place ... and that requires an external force on m_2 . Our 2-body central force formulae apply only to isolated 2-body systems with no such external force (or a constant external force that acts on both bodies equally).
- (b) Now consider the case when m_1 and m_2 are not fixed. (*This* is the isolated 2-body central force system we have been studying!) For the first time on this homework, do *not* make the approximation that one mass is vastly greater than the other. The masses circle each other a constant distance r apart. Find the system’s period, and compare the limit of your result if $m_2 / m_1 \rightarrow \infty$ to the formula you found in part (a).
- (c) What would be the orbital period if the earth were replaced by a star of mass equal to the solar mass, in a circular orbit, with the distance between the sun and star equal to the present earth-sun distance? (The mass of the sun is more than 300,000 times that of the earth.) Express your answer in years.

Problem 5 : The Virial Theorem

The **virial theorem** is a rather well-known theorem in physics. The most common way it is stated is that a particle held in a bound state by an inverse-square central force field (e.g. gravity, the electric force) has kinetic energy equal to half the magnitude of its potential energy. Since the particle is bound, the force is necessarily attractive ($F \sim -1/r^2$), so the potential energy $U \sim -1/r$ is negative. The total energy is thus $E = T + U = \frac{1}{2}|U| - |U| = -\frac{1}{2}|U| = -T \rightarrow$ it’s negative, as it must be for a bound state. This “rule of thumb” that the kinetic energy fills up half of the potential-energy well works wonderfully for many common bound states (planets in orbit, electrons in atoms); it is a really useful fact to know! Let’s examine the virial theorem in greater detail.

- (a) A mass m moves in a circular orbit in the field of a stationary, attractive central force with potential energy $U = kr^n$ (with $kn > 0$ to ensure that the force is attractive). Prove the **virial theorem** : $T = nU / 2$. (The most common appearance of the theorem is for the case $n = -1$, i.e. for a gravity-like potential $U \sim -1/r$. This is the case described in the introductory paragraph. As you see, it gives $T = -U / 2 = \frac{1}{2}|U|$.)
- (b) We can be even more general than that! Let’s derive a form of the theorem that applies to any periodic orbit of a particle, not just a circle. First, find the time derivative of the quantity $G = \vec{r} \cdot \vec{p}$ and, by integrating from time 0 to t , show that $\frac{G(t) - G(0)}{t} = 2\langle T \rangle + \langle \vec{F} \cdot \vec{r} \rangle$, where \vec{F} is the net force on the particle and $\langle f \rangle$ denotes the average over time of any quantity f .
- (c) Explain why, if the particle’s orbit is periodic and if we make t sufficiently large, we can make the left-hand side of this equation as small as we please. That is, the left side approaches zero as $t \rightarrow \infty$. FYI: Notice that the left-hand side can *also* be made zero if we choose $t = n\tau =$ any integer multiple of the orbital period.
- (d) Use this result to prove that if \vec{F} comes from the potential energy $U = kr^n$, then $\langle T \rangle = n\langle U \rangle / 2$, if now $\langle f \rangle$ denotes the time average over a very long time. FYI: Following the FYI from part (c), you can see that the theorem also applies if $\langle f \rangle$ denotes instead the time average over a *cycle* of the periodic orbit. That is the more common interpretation of $\langle T \rangle = n\langle U \rangle / 2$. Taylor’s point is that if you average a periodic quantity over a very long time, you get the cycle average anyway without having to stop your integral at $t =$ exactly $n\tau$.

Problem 6 : Orbit of the Moon

The formulae on page 1 are all tools for obtaining information about one vector: the relative position vector $\vec{r} \equiv \vec{r}_1 - \vec{r}_2$ between two objects. We can obtain its components as functions of time, $r(t)$ and $\phi(t)$, and as functions of each other, $r(\phi)$... all by pretending that \vec{r} describes a fake mass μ moving around a fake fixed source at the origin that produces a central potential $U(r)$.

That's a lot of fakery ... What exactly is the connection between this fake, equivalent 1-body system and the actual, physical 2-body system m_1 and m_2 ? There are two connections: $\vec{r} \equiv \vec{r}_1 - \vec{r}_2$ **IS** the relative position and $U(r)$ **IS** the potential energy between the actual two objects \rightarrow nothing fake about those quantities! To complete our return to reality, let's study the actual positions \vec{r}_1 and \vec{r}_2 of the objects in a 2-body system. To recover those positions from our effective 1-body system, all you need are the *first two equations on page 1*. ☺

Take the Earth as object 1 and the Moon as object 2. The internet has a wealth of information about the Moon's elliptical orbit. Here are two of the many numbers you can find: the perigee distance is 363,000 km and the apogee distance is 405,000 km. (Remember, perigee means "closest approach to the Earth", apogee means "furthest separation from Earth", and when our objects are not points it is implicit that their "positions" are the positions of their centers.) Also, the mass of the Moon is 1.2% of the mass of the Earth: $m_2/m_1 = 0.012$. Using this information, describe the motion of of the Earth's center and the motion of the Moon's center in the **CM frame of the Earth-Moon system**. (No more fake system now! We are asking how the Earth and Moon actually move as viewed by an inertial observer sitting far out in space.)

- Convince yourself that the Earth and Moon both move along ellipses, and figure out the eccentricity of each ellipse.
- What lies at the focal point of the Earth's ellipse? (The Moon? The Earth-Moon CM? Nothing in particular?) Similarly, what lies at the focal point of the Moon's ellipse?
- What is the semi-major axis, a_1 , of the Earth's ellipse? (This tells you how much the Earth's center actually moves as the Earth and Moon orbit around each other.) Express a_1 as a percentage of the Earth's radius of 6,400 km ... you might be surprised at the result. ☺