Physics 326 – Homework #7 due FRIDAY, 1 pm

For today's homework, the **Formula Collection** you may use without proof is the entire set of formulae provided before the first problem of **Discussion 7**. That includes the two equations at the end for r_0 and e in terms of $r_{\min}$ and $r_{\max}$ since you derived them in Homework 6. Everything else, you have to derive here. As in the last homework, use this **Earth Data**: radius of the earth is $R_\oplus = 6.4 \times 10^6$ m; all appearances of the earth's mass $M_\oplus$ will be in the combination $GM_\oplus$ which is equal to $gR_\oplus^2$; use that and the familiar value $g = 9.8$ m/s^2.

Problem 1 : Pluto to Saturn

(a) Before we even introduce this problem, calculate a super-useful general formula for the velocity v_{apse} of an object at an apsidal point of a closed Kepler orbit (i.e. bound by a central force $\vec{F} = -\hat{r}\,\gamma / r^2$): obtain v_{apse} in terms of the corresponding apsidal distance r_{apse}, the parameter r_0 of the orbit, the force constant γ, and the reduced mass μ of the system. Once you have this formula, you may use it in all subsequent problems.

Now on with the story! A research satellite is in Pluto's orbit and needs to be transferred to Saturn's orbit. The distance from the Sun to Pluto is about 40 A.U. while the distance from the Sun to Saturn is about 10 A.U. To save precious funds, the satellite controllers accomplish the switch using a **Hohmann transfer**. If you didn't get to the Hohmann transfer problem in Discussion 7, please have a look at it before you embark on this problem.

(b) Calculate the thrust factors $\lambda \equiv v_{\text{after}} / v_{\text{before}}$ at the launch point from Pluto (λ_L) and at the rendezvous point with Saturn (λ_R). NOTE for this and future problems: Whenever you have a planet orbiting around a star, or a satellite around a planet, the orbiting mass m is so much smaller than the central mass that you may always assume that $\mu \approx m$ (unless otherwise specified, of course).

(c) Show that the satellite's final speed is twice its initial speed.

(d) Calculate the total travel time from launch at Pluto to rendezvous at Saturn.

Problem 2 : GSO = Geo-Synchronous Orbit

An earth satellite is in a circular orbit 250 km above the Earth's surface. Using the most fuel-efficient method available, NASA controllers fire the satellite's thrusters <u>once</u> in order to transfer it to an elliptical **geosynchronous orbit** (GSO) = an orbit whose period matches that of the Earth's rotation. The period of this orbit is thus 24 hours.[1]

(a) Calculate the velocity change, Δv, that is imparted to the satellite to achieve the desired orbit. Remember to include both the magnitude and sign of Δv.

(b) Calculate the apogee distance of the geosynchronous orbit. (There's a decent chance you already calculated it in part (a); if so, emblazon the result with a nice box and an arrow so the grader can find your work. ☺)

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[1] Actually, satellites in GSO, such as many communications and TV satellites, have an orbit of 23 hours 56 minutes = 1 **sidereal day**. This is the period of the Earth's orbit relative to the fixed stars, which provide a fixed inertial reference frame. 24 hours is the period of a **solar day** = the time it takes for the sun to return to the same position in the sky. Since the Earth orbits around the sun, these two periods are not exactly the same, but to simplify your calculator work, just take 24 hours for the period of a satellite in GSO.

Problem 3 : Escape!

A spaceship is "parked" in an elliptic orbit around the Earth. The ship's crew have a long journey ahead of them and they must first escape the Earth's gravitational field.

(a) Calculate the **escape velocity** v_e of the spacecraft as a function of its distance r from the center of the Earth. In case the term is unfamiliar, the escape velocity is the minimum velocity the craft would need at its starting point in order to be able to make it to $r = \infty$ with its engines turned off.

(b) The crew wishes to escape from their elliptical orbit using the most fuel-efficient method possible. As we learned in our Phys 325 study of rocket motion, the amount of fuel used is proportional to the change Δv in the rocket's speed. The crew will apply one impulse from their engines to achieve escape velocity; the question is: at what point on their elliptical parking orbit should they fire their engines to achieve optimal fuel efficiency? Hint: the optimal point is at one of the apses, but you must *show* that this is so, and you must determine which apse (perigee or apogee) is best. This problem is not as trivial as it might seem: in its parked orbit, the ship has the highest velocity at perigee, but it also has the highest *escape velocity* there since perigee puts the ship deeper into the Earth's potential well than at any other point on its orbit. There are thus two competing effects at work, making for an interesting optimization problem.

Problem 4 : A Satellite Experiences a Drag Force

A satellite of mass m moves in the gravitational field of the Earth (mass $M \gg m$), but it is also subject to a linear drag force $\vec{F}_{\text{drag}} = -2\beta m \vec{v}$ where β is the usual positive drag constant.

(a) Show that the satellite's equations of motion can be reduced to the form:

$$\ddot{r} + 2\beta\dot{r} + \frac{GM}{r^2} - \frac{L_0^2 e^{-4\beta t}}{r^3} = 0 \qquad \text{and} \qquad r^2\dot{\phi} = L_0 e^{-2\beta t}$$

where L_0 is a constant that will be assumed to be positive. (If it's negative, the only thing impacted is the sign of $\dot{\phi}$, i.e. whether the satellite moves in the $+\phi$ or $-\phi$ direction.)

(b) Suppose that the drag force is small — i.e. that β is much smaller than all other quantities of the same units — and that the satellite begins in a circular orbit at $t = 0$. By neglecting the terms in $\dot{r}$ and $\ddot{r}$, find an approximate solution for $r(t)$ and $\dot{\phi}(t)$. Your solution should show that small resistance causes the orbit's radius to contract slowly; however, what happens to the speed of the satellite as time increases?

(c) In part (b) we neglected terms in $\dot{r}$ and $\ddot{r}$. This approximation is certainly justified for times near $t = 0$ since the orbit is initially circular, and a circular orbit has both $\dot{r} = 0$ and $\ddot{r} = 0$. Given the solution you obtained, can our neglect of these terms also be justified for times $t > 0$?

Problem 5 : A Wee Bit of Scattering

Let's finish up with a simple problem about the most basic elements of scattering experiments: cross sections and luminosity. These will be discussed in Tuesday's lecture; as this is a short problem, probably best to just wait until Tuesday. If you prefer, you can read Taylor sections 14.1–14.3 to locate the relevant formulae.

The cross section for scattering a certain nuclear particle from a copper nucleus is 2.0 barns. A **barn** is a unit of area that is universally used for describing atomic & subatomic cross-sections; its value is **1 barn = 10^{-28} m²**. If 10^9 beam particles are fired through a copper foil of thickness 10 μm, how many of them are scattered? You will need these values: the density of copper is 8.9 g/cm³ and the atomic mass of copper is 63.5. Recall: the atomic mass gives the mass of an element in amu = atomic mass units where **1 amu = 1.66 × 10^{-27} kg** ≈ the mass of a proton. (The mass of a proton is 1.67 × 10^{-27} kg; the strict definition of an amu is the mass of the ^{12}C nucleus divided by 12, which is not exactly the same due to the binding energy of ^{12}C.)