Physics 326 – Homework #9 due Friday, 1 pm

Here's a summary of our formulae so far related to the inertia tensor:

- $I_{ij} = \int dm\ (\delta_{ij} r^2 - r_i r_j) = \int dm \begin{pmatrix} y^2+z^2 & -xy & -xz \\ \cdot & z^2+x^2 & -yz \\ \cdot & \cdot & x^2+y^2 \end{pmatrix}$
- $\vec{L}^{(B)} = \mathbf{I}^{(B)}\vec{\omega}$ $\quad\forall$ body-fixed ref. pt. B
- Principal Axes $\hat{e}$: $\mathbf{I}\,\hat{e} = \lambda\,\hat{e}$
- $T = \frac{1}{2}\vec{\omega}\cdot\vec{L} = \frac{1}{2}\vec{\omega}^T\mathbf{I}\vec{\omega}$
- $\mathbf{I}^{(B)} = \mathbf{I}_{CM}^{(B)} + \mathbf{I}' = M\left(\delta_{ij}R^{(B)2} - R_i^{(B)}R_j^{(B)}\right) + \int dm\left(\delta_{ij}r'^2 - r_i' r_j'\right)$

You also proved useful symmetry theorems in Discussion 9 for deducing the principal axes (PAs) of an object:

- object is a **lamina** (flat 2D object) → direction perpendicular to the object is a **PA** for rotation around any point in the lamina
- object symmetric under **reflection across a plane** → direction perpendicular to symmetry plane is a **PA** for rotation around any point in the symmetry plane
- object is symmetric under **rotation around an axis** → symmetry axis is a **PA** for rotation around any point on the symmetry axis

Problem 1: Three-Mass Body

A rigid body consists of three masses fastened as follows: m at $(a,0,0)$, $2m$ at $(0,a,a)$, and $3m$ at $(0,a,-a)$.

(a) Find the inertia tensor of this object for rotation around the origin.

(b) Calculate the object's principal axes $\hat{e}_i$ and associated eigenvalues λ_i for rotation around the origin.

(c) Calculate the kinetic energy T of the object if it were rotating with angular velocity ω in three different cases: around the x-axis, the y-axis, and the z-axis (i.e. three separate answers are required). In each case, the rotation axis runs through the origin.

(d) Back when we studied the mass matrix **M** that appears in coupled-oscillation problems, we discussed how to **transform a tensor** from one coordinate system to another. We also showed that when a tensor is transformed into the very special coordinate system that has the tensor's eigenvectors as coordinate axes, the tensor becomes diagonal. Here is a summary of our findings:

Consider two coordinate systems S and S^*. Let **R** be the matrix that transforms vectors from S to S^*: $\vec{v}^* = \mathbf{R}\,\vec{v}$. This is called a **passive transformation**: we are leaving the physical vector unchanged, we are simply re-expressing its components in a different coordinate system. The matrix **R** (or more commonly, its *inverse* $\mathbf{R}^{-1}$ … there's a hint for you!) can be built very easily by writing down the unit vectors of one coordinate system in the other coordinate system. These transformed unit vectors form the *columns* of **R** or $\mathbf{R}^{-1}$, depending on the direction in which you're transforming. Have a look at the Midterm1 formula sheet as a reminder, but better to figure it out from scratch → the key is that (1,0,0) in one system transforms to whatever that first unit vector is in the other system's coordinates (and so on for the other basis vectors). Given that vectors transform as $\vec{v}^* = \mathbf{R}\,\vec{v}$, then tensors **I** transform as follows: $\mathbf{I}^* = (\mathbf{R}^{-1})^T\,\mathbf{I}\,\mathbf{R}^{-1}$. In reverse, you get $\mathbf{I} = \mathbf{R}^T\,\mathbf{I}^*\,\mathbf{R}$. If **R** is a true rotation matrix (which it always is if both S and S^* have orthonormal basis vectors) then it is an "orthogonal matrix" (math language), meaning that it has the very useful property $\mathbf{R}^T = \mathbf{R}^{-1}$.

With this summary in hand, diagonalize the inertia tensor for our 3-mass body via these two steps:

- obtain the rotation matrix **R** that transforms from the $\hat{x}, \hat{y}, \hat{z}$ system to the $\hat{e}_1, \hat{e}_2, \hat{e}_3$ system
- apply this rotation to the tensor **I** to obtain the inertia tensor **I*** of the object in the $\hat{e}_1, \hat{e}_2, \hat{e}_3$ system

Problem 2: Right-Angle Triangle

A thin, flat, uniform metal triangle lies in the *xy* plane with its corners at $(x, y, z) = (1,0,0)$, $(0,1,0)$, and the origin. The triangle's surface density (mass / area) is $\sigma = 24$. (Distances and masses are measured in unspecified units, and the number 24 was chosen to make the answer come out nicely.)

(a) Find the triangle's inertia tensor **I** relative to the origin.

(b) Find the principal moments and a set of orthogonal principal axes through the origin.

(c) Calculate the triangle's inertia tensor $\mathbf{I}'$ for rotations around its center of mass position.

Problem 3 : Spin the Door — Ph.D. Qualifying Exam Problem

A uniform rectangular door of mass M, sides a and $b > a$, and negligible thickness rotates with constant angular velocity ω about the diagonal : the line from one corner of the door to the other. This is all occurring in free space so ignore gravity. Calculate the magnitude of the external torque relative to the door's CM that must be applied to keep the axis of rotation fixed.

Problem 4 : Kick the Right-Angle Triangle

Consider the right-angle triangle you analyzed in problem 2. A blue dot is now painted on the corner of the triangle at the origin. At time $t = 0$, the blue dot is *kicked*: an impulse $k\hat{z}$ is delivered to this corner, where k is a positive constant with units of momentum. Ignore gravity.

(a) Calculate the velocity vector of the blue dot at time $t = 0+$, immediately after the impulse is delivered.

(b) Calculate the velocity vector $\vec{v}(t)$ of the blue dot at all times $t > 0$.

Hint 1: After the impulse is applied, there are no external forces and no external torques on the triangle. What, therefore, are the conserved quantities for its subsequent motion?

Hint 2: Note carefully the direction of $\vec{\omega}\big|_{t=0+}$ from part (a). Is it along a principal axis (PA)? As we showed in lecture, the time-dependence of $\vec{\omega}(t)$ will be greatly simplified if $\vec{\omega}$ starts out parallel to a PA.

Hint 3: Discussion 10, Question 1 will help you a great deal.