Physics 326 – Homework #10 due Friday, 1 pm

All solutions must clearly show the steps and/or reasoning you used to arrive at your result. You will lose points for poorly written solutions or incorrect reasoning. Answers given without explanation will not be graded: ***"No Work = No Points"****. However you may always use any relation on the 1DMath, 3DMath or exam formula sheets or derived in lecture / discussion. Write your* **NAME** *and* **DISCUSSION SECTION** *on your solutions.*

Here's a summary of all our formulae so far related to the inertia tensor:

- $I_{ij} = \int dm\ (\delta_{ij} r^2 - r_i r_j) = \int dm \begin{pmatrix} y^2+z^2 & -xy & -xz \\ \cdot & z^2+x^2 & -yz \\ \cdot & \cdot & x^2+y^2 \end{pmatrix}$
- $\vec{L}^{(B)} = \mathbf{I}^{(B)} \vec{\omega}$ $\forall$ body-fixed ref. pt. B
- Principal Axes $\hat{e}$: $\mathbf{I}\hat{e} = \lambda \hat{e}$
- $T = \frac{1}{2}\vec{\omega}\cdot\vec{L} = \frac{1}{2}\vec{\omega}^T \mathbf{I}\vec{\omega}$
- $\mathbf{I}^{(B)} = \mathbf{I}_{CM}^{(B)} + \mathbf{I}'$
- Euler's Equations: $\vec{\tau} = \dot{\vec{L}} = \dot{\vec{L}}\Big|_{\text{within body}} + \vec{\omega}\times\vec{L}$

$$\tau_1 = I_1\dot{\omega}_1 + (I_3 - I_2)\omega_2\omega_3$$
$$\tau_2 = I_2\dot{\omega}_2 + (I_1 - I_3)\omega_3\omega_1$$
$$\tau_3 = I_3\dot{\omega}_3 + (I_2 - I_1)\omega_1\omega_2$$

- Free Symmetric Top : precession of $\vec{\omega}$ is $\vec{\Omega}^* = \left(\frac{I_3}{I_1} - 1\right)\omega_3 \hat{e}_3$ body, $\vec{\Omega} = \frac{\vec{L}}{I_1}$ lab; $\vec{L}, \vec{\omega}, \hat{e}_3$ always coplanar

You also proved several extremely useful <u>symmetry theorems</u> in Discussion 9; those are at your disposal too!

Problem 1 : Practice with Euler's Equations

Note: Please <u>use Euler's three equations</u> to solve these, even if you could solve them some other way (e.g. by directly using the "master" equation on which they are based, $\vec{\tau} = d\vec{L}/dt$). The entire point of this problem is to simply become *familiar* with the *structure* of these equations. They can all be solved by taking <u>linear combinations</u> of the equations (or using just one of them) to construct the quantity you are studying in each part.

(a) A rigid body is rotating freely, subject to zero torque. Use Euler's equations to prove that the magnitude of the angular momentum is constant. Hint: you can just show L^2 is constant, and L^2 has a very nice form in the body-system used for Euler's equations! Write down its derivative, dL^2/dt, then manipulate Euler's equations (linear combination!) to build that dL^2/dt expression … it comes out really nicely. ☺

(b) In much the same way, show that the kinetic energy of rotation, $T_{rot} = \frac{1}{2}\vec{L}\cdot\vec{\omega}$, is constant.

(c) Consider a lamina rotating freely (no torques) about a point O in the lamina. Use Euler's equations to show that the component of $\vec{\omega}$ in the plane of the lamina has constant magnitude.
(i.e. If you choose $\hat{e}_3$ as perpendicular to the lamina, you must show that the time derivative of $\omega_1^2 + \omega_2^2$ is zero.)
Hint: A pure lamina is completely flat : it has *no size* in the direction perpendicular to its surface. This causes an additional simplification in the inertia tensor beyond certain off-diagonal elements going to zero → it imposes a strict relationship between the diagonal elements, i.e., one of them can be written in terms of the other two. You'll need this relationship; it's easy to figure out.

(d) Consider an axisymmetric object rotating freely (i.e. no torques) about a point O on its axis of symmetry. What do Euler's equations tell us about the time-dependence of the component of $\vec{\omega}$ along the object's axis of symmetry?

Problem 2 : Angles for a Free Symmetric Top

In our study of a torque-free symmetric top, we found the exceedingly important relation that the vectors $\vec{L}$, $\vec{\omega}$, and $\hat{e}_3$ always remain <u>coplanar</u>. (Recall that $\hat{e}_3$ is the axis of symmetry of the top.) This coplanarity provides a crucial link between the body frame, where $\hat{e}_3$ is fixed, and the lab frame, where $\vec{L}$ is fixed. In addition, the <u>angles</u> between these three vectors remain constant throughout the object's motion.

Just FYI: for basically all "free top" problems, the quantities you must be given to make the system solvable are

- the top's principal moments I_i, or enough information about the top to calculate them
- some information about $\vec{\omega}$, e.g. the components ω_3 and $\omega_{12} = |\omega_1\hat{e}_1 + \omega_2\hat{e}_2|$ (which are constants of motion for an axisymmetric top), the magnitude ω and some angle, or some initial value $\vec{\omega}\big|_{t=0}$

(a) Calculate the angle α between the vectors $\vec{L}$ and $\hat{e}_3$ in terms of I_1, I_3, ω_{12}, and/or ω_3.

(b) Do the same for the angle β between the vectors $\vec{\omega}$ and $\hat{e}_3$.

(c) I throw a thin, flat, uniform circular disc into the air so that it spins with angular velocity ω about an axis that makes an angle β with the symmetry axis of the disc. What is the precession (rotation) frequency of the disc's symmetry axis around the angular momentum vector, as seen by me (lab frame)? Amazingly, the answer depends only on ω and $\sin\beta$.

Problem 3 : The Space Station from 2001

An axially symmetric space station (e.g. a torus, as depicted in the movie "2001", or a cylinder) is floating in free space. It has rockets mounted symmetrically on opposite sides. The rockets are continuously firing so as to exert a constant torque τ around the station's axis of symmetry, $\hat{e}_3$.

(a) As it happens, the station's rotation is not aligned with its symmetry axis → at time $t = 0$, the rotation vector is $\vec{\omega}\big|_{t=0} = \omega_{20}\hat{e}_2 + \omega_{30}\hat{e}_3$, where ω_{20} and ω_{30} are constants. Solve Euler's equations exactly for $\vec{\omega}(t)$ in the body frame using this initial condition. If you need it, a hint is provided after the last problem about how to solve the coupled differential equations you will obtain.

(b) Describe the motion of the station in words, as seen by an inertial observer floating outside the station.

Problem 4 : Adding a Mountain to a Spherical Planet

Imagine that this world is a perfectly rigid uniform sphere and is spinning about its usual axis at its usual rate. A huge mountain of mass 10^{-8} earth masses is now added at colatitude 60°, where "colatitude" means "angle with respect to the north pole". (The Earth-geography latitude of this mountain would thus be 30° N.) The mountain ruins the earth's perfect symmetry (how sad) and so causes the Earth to begin free precession, like any other axisymmetric body. Assuming that <u>no torques</u> affect the Earth's motion, how long will it take the North Pole (defined as the northern end of the diameter along $\vec{\omega}$) to move 60 km <u>along the Earth's surface</u> from its current position? Take the Earth's radius to be 6400 km. Note: the input values you've been given have been specified to at most *two significant digits*, so it makes *no* sense to obtain your answer to any greater level of precision than that. ☺

Math Hint for Problem 3(a) : The Space Station

You will encounter coupled equations of this type:

$$\begin{aligned} \dot{f}_1(t) &= -K(t) f_2(t) \\ \dot{f}_2(t) &= K(t) f_1(t) \end{aligned}.$$

If that coefficient $K(t)$ was a *constant*, K, you would immediately know the form of the functions: one of them is a sine and one of them is a cosine. They may have some overall phase shift or some amplitudes of common magnitude to satisfy the boundary conditions, e.g. $-6\sin(\omega t - 45°)$ and $6\cos(\omega t - 45°)$ … but whatever the details, you know f_1 and f_2 are sinusoidal functions of time that are 90° out of phase with each other. There is no other pair of functions that will give you "derivative of f_1 is –*blah* f_2 and derivative of f_2 is +*blah* f_1". The only unfamiliar aspect of problem 3(a) is that *blah* is a function of time, not a constant. Well you can *still* solve the equations by guessing well. Will a sine and a cosine still work? Absolutely : even when *blah* is time-dependent, there is still no other pair of functions that gives you "derivative of f_1 is –*blah* f_2 and derivative of f_2 is +*blah* f_1". You are accustomed to the solution forms $A\sin(\omega t + \phi)$ and $A\cos(\omega t + \phi)$… you just have to rethink them a little bit. You need some additional time-dependence somewhere, to accommodate that $K(t)$ coefficient ... where shall we put it? How about in the argument of the sinusoidal functions? ➔ ωt is a bit too simple, that's all, so try replacing it with some unknown function of time, $g(t)$: try $A\sin[g(t)]$ instead of $A\sin[\omega t]$. Plug forms like that for f_1 and f_2 into your differential equations and you will quickly see what $g(t)$ has to be.