Physics 326 – Homework #12 Due FRIDAY, 1 pm

All solutions must clearly show the steps and/or reasoning you used to arrive at your result. You will lose points for poorly written solutions or incorrect reasoning. Answers given without explanation will not be graded: ***"No Work = No Points"****. However you may always use any relation on the 1DMath, 3DMath or exam formula sheets or derived in lecture / discussion. Write your* **NAME** *and* **DISCUSSION SECTION** *on your solutions.*

Discussion 12 summarized the axioms of General Relativity, including the (non-axiomatic) **Schwarzschild metric** that we are using as our example of curved spacetime. It is the metric describing the spacetime curvature around the most familiar type of mass: a spherically-symmetric non-rotating mass M.

$$\boxed{d\tau^2 = dt^2\left(1-\frac{2M}{r}\right) - \frac{dr^2}{\left(1-\frac{2M}{r}\right)} - r^2 d\phi^2}$$ in **natural units** : $M = \frac{G}{c^2} M_{\text{kg}}$ and $t = c\, t_{\text{sec}}$

In natural units, the annoying constants G and c vanish and the r, M, t, and τ variables all come out in the same units : meters. One of the two main challenges of GR is figuring out how to use the metric to solve problems! The other challenge is how to find the metric for different mass distributions from Einstein's field equations … but that is an advanced topic treated in graduate courses. (Note that Einstein himself found it challenging: Schwarzschild, not Einstein, was the first one to come up with such a solution. ☺)

Problem 1 : The Schwarzschild Radius = The Event Horizon of a Black Hole

(a) Here's a small problem but one you *must* do at least once: show that the Schwarzschild radius $R_S \equiv 2M$ is precisely the radius at which the escape velocity is c. Astonishingly, you do not need the machinery of GR at all to obtain this result, a purely Newtonian calculation gives $R_S \equiv 2M$ … which is so extraordinary that one wonders if it isn't an accident. ☺

(b) The density of normal air is about 1 kg / m^3. Is it possible to create a black hole out of air? Our "sci fi" image of black holes makes it seem absurd, but it's not impossible at all. Figure out how big a black hole's radius would have to be if it were made out of air, i.e. with a uniform mass density of no more than 1 kg / m^3. Compare your result to some astronomical scales: the sun, the solar system, the Milky Way galaxy, ...

Problem 2 : The GR Lagrangian

All aspects of GR flow from the metric. The dynamics of gravitational systems – i.e. the motion of particles in a gravitational field – is determined via this supremely elegant axiom:

> The paths that particles follow are the **geodesics** – the "straight lines" – of the spacetime geometry through which they are passing. Specifically, they are the paths that **maximize** the amount of proper time $\int d\tau$ between any two endpoints. This is sometimes called the **Principle of Maximal Aging**.

GR thus motivates the use of a **variational principle** for mechanics in a supremely natural way!

- In GR, the paths that physical systems take through spacetime are those that maximize $\int d\tau$.
- In Lagrangian mechanics, the paths that systems take are those that minimize $\int L\,dt$.

Since the Euler-Lagrange equations do not care if you are maximizing or minimizing your integral, the function $\boxed{L = d\tau / dt}$ provides an "operational Lagrangian" for GR: if we apply the Euler-Lagrange equations to L, we obtain the equations of motion for an object moving in the metric dτ as a function of time.

We can also recast the Principle of Maximal Aging more precisely into the Principle of Least Action: in the Newtonian limit,

$$\text{Maximizing } \int d\tau \qquad \text{must be the same as} \qquad \text{Minimizing } \int L_{SI}\ dt = \int (T-U)\ dt$$

The result that, in SI units, the Lagrangian of GR (and SR!) is $\boxed{L_{SI} = -mc^2 d\tau / dt}$. $L = d\tau / dt$ works just fine for obtaining equations of motion; $L_{SI} = -mc^2 d\tau / dt$ is the quantity that maps onto *T–U* in the Newtonian limit.

(a) Let's see if it our formula *does* reproduce the familiar Lagrangian *L* = *T* – *U!* Working in SI units for a change, calculate $L_{SI} = -mc^2 d\tau / dt$ from the Schwarzschild metric, then apply Taylor approximations to restrict ourselves to the regime where *L* = *T* – *U* works:

(1) at slow speeds $v \ll c$ where Newtonian mechanics works (no special relativity needed), and
(2) in weak gravitational fields $GM / rc^2 \ll 1$ where Newton's $F = GMm / r^2$ gravitational force law works.

Show that, when approximated to lowest non-vanishing order in both of the small quantities v / c and GM / rc^2, the Schwarzschild metric and the Principle of Maximal Aging do indeed reproduce $L = T - U$. Remember from Phys 325 that we have some freedom in the way we write a Lagrangian:

- Since only the *derivatives* of *L* appear in the Euler-Lagrange equations, you can add or drop *constant terms* from any Lagrangian without changing the equations of motion.
- Since *L* appears in *all terms* of the Euler-Lagrange equations, you can also multiply any Lagrangian by a *constant factor* without changing the equations of motion. Since a particle's rest mass is a scalar (a frame-independent constant), the change from $L = -d\tau / dt$ to $L = -mc^2 d\tau / dt$ doesn't affect the equations of motion at all, it will just help you to recognize *T–U* when you find it.

(b) Many newcomers to GR wonder why the geodesics of spacetime maximize $\int d\tau$ rather than minimizing it. The reverse is true in spatial geometries, where geodesics are the *shortest* paths between two points, not the longest ones. There are two ways you can see that this is true. The first one is the uniqueness of geodesics: by definition, the geodesic path between two points *extremizes* the total distance (as defined by the metric); whether the extremum in question is a *maximum* or a *minimum* depends on which if those choices gives you a *unique* path between two points. In the flat spatial world, it is clear that demanding the *minimum* distance between two points gives us a unique answer = a straight line; in contrast, there are an infinite number of paths you can draw that have an infinite = maximal length. In the world of spacetime, where proper time provides the metric, the reverse is true. To see this, do the following: (You can fit all the diagram questions on one figure if you like.)

(b1) Draw a Minkowski diagram with time *t* going upwards and some spatial coordinate *x* pointing to the right (the usual SR convention for these sketches).

(b2) Mark on the diagram two very simple endpoints: a starting point at the origin (*t*,*x*) = (0,0), and an ending point at the same place (*x* = 0) but 10 seconds later (*t* = 10).

(b3) Write down the metric for **flat spacetime**, i.e. the metric in the absence of gravity. This is just the familiar metric of SR = the definition of proper time. Use only terms in *dt* and *dx* (ignore *dy* and *dz* for brevity), then factor out *dt*. You should get this in SI units: $d\tau = dt\sqrt{1-(\dot{x}/c)^2} = dt\sqrt{1-\beta^2}$ (using the SR notation $\beta \equiv v / c$).

(b4) First find a path or paths that maximize $\int d\tau$ for between the two endpoints you drew. You can make the particle do anything, it just has to start at $(t,x) = (0,0)$ and end at $(t,x) = (10,0)$ … and any motion it makes will affect its proper time via $d\tau = dt\sqrt{1-\beta^2}$. If you can find more than one path that maximizes $\int d\tau$, draw in more than one to show that the result is *not unique*. HINT: β can only take on certain values ... use that fact to *first* figure out the maximum & minimum possible values of $\Delta\tau$. That's all you need to figure out the paths to draw, so there is NO need to perform any variational calculations.

(b5) Now find a path or paths that minimize $\int d\tau$ between your two endpoints. Again, if you can find more than one such path, draw in more than one to show non-uniqueness.

(b6) If all went well, only one of the last two bullets produced a unique extremal path. Does this unique path maximize or minimize proper time?

(b7) What is the particle *doing* on the unique path you found? Does this motion (or lack thereof) correctly describe the physical behaviour of a particle in free space with no forces acting on it? (Remember: the flat metric $d\tau = dt\sqrt{1-\beta^2}$ you've been using has no gravity in it, or any other forces.)

(b8) To complete our understanding of the Principle of Maximal Aging, we must understand that last word: "Aging". If all went well, you should have at least three paths drawn on your figure. For each path (well, for at least three of them ☺), indicate on the figure how much older each particle is in seconds when it reaches the endpoint compared with when it left the starting point.

I hope this exercise has cemented your intuition about the Principle of Maximal Aging … and about the variational way of thinking about mechanics in general!

Problem 3 : Conserved Energy & Radial Fall

You have all the tools you need to calculate the geodesics for any metric! It's just variational calculus: write the "maximize $\Delta\tau$" principle in Lagrangian form and apply the Euler-Lagrange equations. *Solving* those equations is another matter, however. With $d\tau$ equal to a big square root of a complicated argument, the GR equations of motion are quite unpleasant. An excellent alternate strategy is to find **conserved quantities** for the geodesics you seek. How? Look for **cyclic coordinates** q_i that do not explicitly appear in the Lagrangian. Recap:

- if $\dfrac{\partial L}{\partial q_i} = 0$ then $\dfrac{\partial L}{\partial \dot{q}_i}$ is conserved
- if $\dfrac{\partial L}{\partial t} = 0$ then $H \equiv \dfrac{\partial L}{\partial \dot{q}_i}\dot{q}_i - L$ is conserved

In this problem, we will study the trajectory of a particle falling into a black hole from infinity. We will consider radial motion only, so we have only on generalized coordinate, r, and the metric simplifies to $d\tau^2 = dt^2\left(1-2M/r\right) - dr^2/\left(1-2M/r\right)$.

(a) The Schwarzschild metric (and so the associated Lagrangian) has no explicit dependence on time, so the Hamiltonian H is conserved. This is also a situation in which the Hamiltonian is equal to the system's energy, since we are using natural coordinates (i.e. no time-dependent constraints … no constraints of any kind in fact). Calculate the Hamiltonian to show that the energy of a our mass m particle falling into the gravity well of a mass M is given by $\boxed{E = mc^2\left(1-\dfrac{2GM}{rc^2}\right)\dfrac{dt}{d\tau}}$ in SI units.

☞ Note: calculation of energy via the Hamiltonian is the *only* place where it matters if you use L or L_{SI}. If you use the reduced version L of the Lagrangian, you will obtain a perfectly useful constant of motion that is *proportional* to the particle's energy, it just won't be energy exactly. Now that we have our full formula, we can shorten it to $\boxed{e = \dfrac{dt}{d\tau}\left(1-\dfrac{2M}{r}\right)}$. $e \equiv E/mc^2$ is a nice dimensionless constant proportional to the particle's energy, and we are back in natural units. ☺

(b) Use the fact that energy is conserved to determine the speed dr/dt as a function of r. The constant of motion e of the particle will appear in your result (to be determined by the initial conditions).

(c) Suppose the particle started <u>at rest at $r = \infty$</u>. What is the constant of motion e in this situation? Use this value of e to simplify your velocity expression from (b), and also use it for part (d).

What you've calculated is "coordinate speed": dr/dt in the (r,t) coordinate system of the Schwarzschild metric. The observer who uses these coordinates is basically an observer at infinity whose coordinates are unaffected by strange mass-induced spacetime curvatures. Taylor & Wheeler have a colorful name for this observer: the "**Bookkeeper**" = an accountant who provides a reference set of clocks and rulers. The Bookkeeper, being at infinity, never *directly observes anything*. Consider the particle from part (c) that is traveling at dr/dt $\rightarrow$ once it leaves infinity, the Bookkeeper cannot observe it directly since he is never *there*.

Now let's consider **Local Observers** = observers who are *at* the location of our moving particle as it passes by. What do they see? The Bookkeeper stays at infinity, so his clock time t is never affected by gravitational time dilation and his radial ruler r is unaffected by reduced-radius effects[1]. A local observer is one who is sitting at radius r with her clocks and rulers and only measures things that occur <u>at or very near her location</u>. This observer is *in* the curved spacetime of the gravity-well and it *does* affect her clocks and rulers. When she measures a time interval with her wristwatch, she measures $dt_{\text{local}} = d\tau\big|_{\text{fixed } r,\phi}$, which is not the Bookkeeper's dt; when she measures a radial distance with her ruler, she measures $dr_{\text{local}} = d\sigma\big|_{\text{fixed } t,\phi}$, which is not the Bookkeeper's dr.

(d) Calculate the velocity $dr_{\text{local}}/dt_{\text{local}}$ seen by local observers who only make their speed measurement when the particle passes right by their location.

(e) Make a plot of speed vs coordinate-distance r showing both of the speeds you calculated: $|dr/dt|$ and $|dr_{\text{local}}/dt_{\text{local}}|$. You may be surprised at how different they are! Be sure to indicate where the speed of light is on your plot's vertical axis. Also, remember that our formalism breaks down below $r = 2M$ (the event horizon), so your plot should not extend below that radius.

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[1] If you are doing this homework before this week's first lecture, don't worry about reduced-radius $\rightarrow$ it will be explained.