

Physics 326 – Homework #13 Part A

due **WEDNESDAY May 4, 5 pm**

All you need for GR calculations is the Schwarzschild metric. The Hamiltonian = the particle's total energy (rest + kinetic + potential energy) is also useful because it is conserved. You derived it on the last homework, so you can use it without proof this week:

$$d\tau^2 = dt^2 \left(1 - \frac{2M}{r}\right) - \frac{dr^2}{\left(1 - \frac{2M}{r}\right)} - r^2 d\phi^2 \quad \text{and} \quad e = \frac{dt}{d\tau} \left(1 - \frac{2M}{r}\right) \quad \text{where } e = \frac{E_{\text{TOTAL}}}{mc^2} \text{ in SI units}$$

Problem 1 : The Path of Light

(a) Light follows a very special geodesic: the **null geodesic** $d\tau = 0$. In words, proper time is slowed down to a complete stop for light. One way to prove this is using one of the defining properties of light: it has zero rest mass. Use one of the formulas at the top to show that $m = 0$ implies $d\tau = 0$.

(b) Given that $d\tau = 0$ for light, calculate the speed of light as a function of r for a photon falling radially toward the center of a black hole. Calculate two versions of this speed: the “Bookkeeper” speed $|dr/dt|$ and the local speed seen by stationary local observers measuring local radial distance with their rulers and local time with their wristwatches as the photon passes right by them. Plot these two speeds vs r/M , being careful to label which is which, and to label the one important value on the horizontal axis.

Problem 2 : Time of Radial Fall

Alice is in a spaceship of mass m and constant energy e that falls under gravity (no engines) toward the center of a spherical mass M along a radial path. Last week, you used the energy formula and a bit of algebra to obtain

the Bookkeeper speed of such a radially-falling mass: $\dot{r} = -\left(1 - \frac{2M}{r}\right) \sqrt{1 - \left(\frac{1}{e}\right)^2 \left(1 - \frac{2M}{r}\right)}$. To simplify your calculations, let's use $e = 1$.

(a) What does $e = 1$ tell us about the spaceship's initial conditions?

(b) How much Bookkeeper time does it take for the spaceship to fall to the black hole's event horizon?

Calculate a formula for $\bar{t} \equiv t/M$ as a function of $\bar{r} \equiv r/M$. (Just start calculating t and a function of r , then as you construct your integral, notice how M can be completely removed from the problem by rescaling t and r .)

It requires a nasty integral, but one that wolframalpha can do just fine! To interpret your result, let's set our clock to $t = 0$ at somewhere finite, say at $r = 5M$. Plot $\bar{t}(r) - \bar{t}(r = 5M)$ versus \bar{r} over the range $\bar{r} = 2$ to 8 .

(c) How much proper time does it take for the spaceship to fall to the black hole's event horizon?

Calculate a formula for $\bar{\tau} \equiv \tau/M$ as a function of $\bar{r} \equiv r/M$, then plot $\bar{\tau}(r) - \bar{\tau}(r = 5M)$ versus \bar{r} again over the range $\bar{r} = 2$ to 8 . Finally, in words: what exactly does the proper time difference $\tau(r=2M) - \tau(r=5M)$

represent? i.e. What does it mean physically?

Problem 3 : Say Hello to the EM Lagrangian

The Lagrangian for GR and SR is $L = -mc^2 d\tau / dt$ in SI units. This does not have the form $L = T - U \rightarrow$ not all Lagrangians do! (What you showed last week is that it *reduces* to T-U in the Newtonian limit of slow speed and weak gravity.) The Lagrangian that describes the motion of a non-relativistic particle of mass m and charge e in an electromagnetic field also does not have T-U form: it is

$$L_{EM} = \frac{1}{2} m \dot{\vec{r}}^2 - e(V - \dot{\vec{r}} \cdot \vec{A}).$$

Here $\{V, \vec{A}\}$ are the {scalar,vector} potentials of E&M that give the fields via the relations $\vec{E} = -\vec{\nabla}V - \partial\vec{A}/\partial t$ and $\vec{B} = \vec{\nabla} \times \vec{A}$. We are using “ e ” for the particle’s charge since q could cause confusion with “generalized coordinate”. Obviously we can write this as $L_{EM} = T - e(V - \dot{\vec{r}} \cdot \vec{A})$, which *looks* like T-U form, but it really isn’t. Let’s explore.

(•) Obviously you will be profoundly unhappy unless you verify for yourself that this unusual Lagrangian does indeed reproduce the E&M force law, $\vec{F} = e(\vec{E} + \vec{v} \times \vec{B})$! Use Cartesian coordinates (x, y, z) for the particle’s position, then write down and expand the Euler-Lagrange equation for just one coordinate, x . If you can show that this equation reduces to $m\ddot{x} = F_x = e(E_x + \dot{y}B_z - \dot{z}B_y)$, it will be true for y and z as well. This derivation is not for points since it is done in Taylor section 7.9, but please do try it yourself!

(a) L_{EM} certainly *looks* like a typical Lagrangian of the form $L = T - U$, with a velocity-dependent potential energy $e(V - \dot{\vec{r}} \cdot \vec{A})$. However, this is *not* a potential energy and should even not be given the letter U ... we can call it U^* . Why all the caution? The defining property of a potential energy U is that it describes a force via the standard relation $\vec{F} = -\vec{\nabla}U$. Well this one doesn’t! Calculate the gradient of $U^* = e(V - \dot{\vec{r}} \cdot \vec{A})$ and show that $-\vec{\nabla}U^*$ is *not* the Coulomb-Lorentz force $\vec{F} = e(\vec{E} + \vec{v} \times \vec{B})$.

Hint 1: Perform your calculation in Cartesian components with index notation: $-\vec{\nabla}U^* = -\hat{x}_i \partial U^* / \partial x_i$

Hint 2: The easiest way to show that $-\vec{\nabla}U^*$ is *not* $e(\vec{E} + \vec{v} \times \vec{B})$ is to find one example that disproves it.

Specifically, find a vector-potential \vec{A} that produces *no magnetic field* (i.e. an \vec{A} that has no curl) but does have some derivative: one that is sufficient to make an appearance in $-\vec{\nabla}U^*$ and cause it to *not* match the force $e\vec{E}$.

☞ **Take-away message:** L_{EM} is another example of a Lagrangian that does *not* follow the form $L = T - U$.

(b) A particle of mass m and charge e moves in a magnetic field produced by a constant current I flowing in an infinite straight wire that lies along the z -axis. This B -field is given by the potentials $A_s = A_\phi = V = 0$ and $A_z = -(\mu_0 I / 2\pi) \ln(s)$ in the cylindrical coordinate system (s, ϕ, z) . Find the Lagrangian of the particle, note that ϕ and z are cyclic coordinates, and find the corresponding conserved quantities p_ϕ and p_z .

☞ **Crucial point:** The conserved p_z you just found is the generalized momentum conjugate to z ... and it is NOT the z -component of mechanical momentum $(mv)_z$! \rightarrow For this important E&M Lagrangian, we must be very careful to distinguish the two DIFFERENT concepts of “momentum”:

- the canonical momentum \vec{p} obtained from $p_i = \partial L / \partial \dot{x}_i$, and
- the mechanical momentum $m\vec{v} \rightarrow$ Do NOT call this “ p ” for the rest of this problem, call it “ mv ”!

☞ **Jargon check:** The terms “canonical momentum”, “generalized momentum”, and “momentum conjugate to q_i ” all mean exactly the same thing: $p_i \equiv \partial L / \partial \dot{q}_i$.

(c) Interpret the two conserved quantities you found by expressing them in terms of components of the following physical quantities: the particle's linear momentum $m\vec{v}$, the particle's angular momentum \vec{l} , its charge e , and/or \vec{A} itself. Remember: to avoid massive confusion do NOT USE THE LETTER “ p ” for the combination “ mv ”! We are reserving “ p ” for canonical momentum.

(d),(e)-combined for clarity

To explore these two definitions of momentum further, consider the E&M Lagrangian and Hamiltonian for any electromagnetic potentials V and \vec{A} (not just the ones from part b). Go to Cartesian coordinates $q_i = (x, y, z)$ and show that:

- the relationship between mechanical and canonical momentum is $m\vec{v} = \vec{p} - e\vec{A}$;
- the Hamiltonian for a charge e of mass m in an EM field is $H = \frac{1}{2m}(\vec{p} - e\vec{A})^2 + eV$.

☞ **The notorious substitution** : The relations you just found lead to a confusing “trick” that is invoked in other areas of physics, notably quantum mechanics. In various QM texts, you will see the author take a regular Hamiltonian, $H = T + U = p^2 / 2m + U$, then make this substitution: they replace \vec{p} with $\vec{p} - e\vec{A}$ to “add electromagnetism” to their equations of motion. This substitution is often done with little or no explanation, or by saying it is “familiar from classical mechanics”. (eye-roll) Now you know where it comes from ... and you also know that the “ p ” in question is canonical momentum, NOT mechanical momentum.

(f) Finally, rewrite your Hamiltonian in this way: *remove* the canonical momentum \vec{p} and *replace* it with the combination $m\vec{v} + e\vec{A}$. Since we *know* what $m\vec{v}$ is — it's the particle's actual, mechanical momentum — we can now interpret this Hamiltonian! Use your knowledge of the *work done by a magnetic field* to show that your E&M Hamiltonian is, in fact, the total (kinetic + potential) energy of a charge in an electromagnetic field.

☞ **A Vital Message**: The Hamiltonian $H_{EM} = (\vec{p} - e\vec{A})^2 / 2m + eV$ can lead to *enormous* confusion.

(I have seen it happen, e.g. throughout a 2-week research workshop with numerous theorists in attendance.) The key point: **the “ p ” in this Hamiltonian is NOT the particle's momentum mv !** People constantly assume that it is, then go on to believe that the $-e\vec{A}$ term added to \vec{p} is some sort of “field momentum”. As you saw in (f), our Hamiltonian is nothing more than $H_{EM} = mv^2 / 2 + eV = T + U$ of the charged particle; it does not contain *any trace whatsoever* of the momentum or energy of the EM fields. If field energy plays a role, V and/or \vec{A} will be time-dependent and lead to $\partial H / \partial t \neq 0$, i.e. a non-conserved Hamiltonian H . To restore conservation of energy, a different expression is needed that includes the energy in both the particles and the fields. Chapter 8 of Griffiths' “*Introduction to Electrodynamics*” contains an excellent discussion of energy and momentum conservation in E&M ... though not in Hamiltonian or Lagrangian form. To go further, you must study the Lagrangian formulation of field theories, and that is well beyond the scope of our course.