# Physics 435 – Equation Sheet for Midterm 3

## Electrostatics

$$\vec{F} = q\vec{E} \qquad \vec{\mathscr{r}} = \vec{r} - \vec{r}'$$

$$\vec{E}(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\int \frac{dq}{\mathscr{r}^2}\hat{\mathscr{r}}$$

$$V(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\int \frac{dq}{\mathscr{r}}$$

$dq \to \rho\, d\tau'$ (volume)
$dq \to \sigma\, da'$ (area)
$dq \to \lambda\, dl'$ (line)

$$\vec{E}(\vec{r}) = -\nabla V(\vec{r})$$

$$V(\vec{r}) = -\int_{\infty}^{\vec{r}} \vec{E}(\vec{r})\cdot d\vec{l}$$

$$W = \frac{\varepsilon_0}{2}\int \left|\vec{E}\right|^2 d\tau$$

## Capacitance / capacitors

$$C = \frac{Q}{V}$$

$$C = \frac{A\,\varepsilon_0}{d}$$

## Divergence / Stokes Theorem

$$\underset{\text{surface}}{\oint} \vec{E}\cdot d\vec{a} = \underset{\text{volume}}{\int} (\nabla\cdot\vec{E})\,d\tau$$

$$\underset{\text{surface}}{\int} (\nabla\times\vec{B})\cdot d\vec{a} = \underset{\text{contour}}{\oint} \vec{B}\cdot d\vec{l}$$

## E-fields for simple charge configurations

$\vec{E} = \frac{Q}{4\pi\varepsilon_0 r^2}\hat{r}$ (point charge, at origin)

$\vec{E} = \frac{\lambda}{2\pi\varepsilon_0 s}\hat{s}$ (line charge, along z axis)

$\vec{E} = \frac{\sigma}{2\varepsilon_0}(\pm\hat{z})$ (plane charge, in xy plane)

## Laplace's Equation

$$\nabla^2 V = -\frac{\rho}{\varepsilon_0} \;\to\; \nabla^2 V = 0 \text{ (if } \rho = 0)$$

**Separation of variables for $\nabla^2 V = 0$:**

Cartesian:

$$V(x,y) = \sum_{n=0}^{\infty}\left(A_n e^{k_n x} + B_n e^{-k_n x}\right)(C_n \sin k_n y + D_n \cos k_n y)$$

Spherical:

$$V(r,\theta) = \sum_{\ell=0}^{\infty}\left(A_\ell r^\ell + \frac{B_\ell}{r^{\ell+1}}\right)P_\ell(\cos\theta)$$

## Maxwell's Equations
(without displacement current)

differential form:

$$\nabla\cdot\vec{E} = \frac{\rho}{\varepsilon_0} \qquad \nabla\cdot\vec{B} = 0$$

$$\nabla\times\vec{E} = -\frac{d\vec{B}}{dt} \qquad \nabla\times\vec{B} = \mu_0\vec{J}$$

integral form:

$$\oint \vec{E}\cdot d\vec{a} = \frac{Q}{\varepsilon_0} \qquad \oint \vec{B}\cdot d\vec{a} = 0$$

$$\oint \vec{E}\cdot d\vec{l} = -\frac{d}{dt}\int \vec{B}\cdot d\,\vec{a} \qquad \oint \vec{B}\cdot d\vec{l} = \mu_0 I$$

# Physics 435 – Equation Sheet for Midterm 3

## Boundary conditions near sheet charge

$$\vec{E}_{\text{above}} - \vec{E}_{\text{below}} = \frac{\sigma}{\varepsilon_0}\hat{n}$$

$$\frac{dV_{\text{above}}}{dn} - \frac{dV_{\text{below}}}{dn} = -\frac{\sigma}{\varepsilon_0}$$

$$B^{\perp}_{\text{above}} = B^{\perp}_{\text{below}}$$

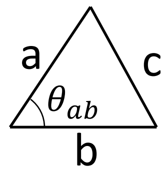
$$B^{\parallel}_{\text{above}} - B^{\parallel}_{\text{below}} = \mu_0\left(\vec{k}\times\hat{\ell}\right)$$

$$\vec{A}^{\perp}_{\text{above}} = \vec{A}^{\perp}_{\text{below}}$$

$$\frac{\partial\vec{A}_{\text{above}}}{\partial n} - \frac{\partial\vec{A}_{\text{below}}}{\partial n} = -\mu_0\vec{k}$$

## Law of cosines

$$c^2 = a^2 + b^2 - 2\,a\,b\cos\theta_{ab}$$




## Multipole expansion

$$V(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\sum_{n=0}^{\infty}\frac{1}{r^{n+1}}\int (r')^n P_n(\cos\alpha)\,\rho(\vec{r}\,')d\tau'$$

$$V(\vec{r}) = \frac{Q}{4\pi\varepsilon_0 r} + \frac{\vec{p}\cdot\hat{r}}{4\pi\varepsilon_0 r^2} + \cdots \text{(these are the monopole + dipole terms)}$$

$$\vec{p} \equiv \int \vec{r}\,'\,\rho(\vec{r}\,')d\tau'$$

## Dipoles

$$\vec{p} = \alpha\vec{E}$$

$$\vec{\tau} = \vec{p}\times\vec{E}$$

$$\vec{F} = (\vec{p}\cdot\nabla)\vec{E}$$

## First few Legendre polynomials

$$P_0(x) = 1$$
$$P_1(x) = x$$
$$P_2(x) = (3x^2-1)/2$$
$$P_3(x) = (5x^3-3x)/2$$

## Polarization

$$\vec{P} = d\vec{p}/d\tau$$

$$V_{\text{di}}(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\oint\frac{\sigma_b\,d\vec{a}\,'}{\mathscr{r}} - \frac{1}{4\pi\varepsilon_0}\int\frac{\rho_b}{\mathscr{r}}d\tau'$$

$$\sigma_b = \vec{P}\cdot\hat{n}$$

$$\rho_b = -\nabla\cdot\vec{P}$$

## The displacement

$$\vec{D} = \varepsilon_0\vec{E} + \vec{P}$$

$$\rho = \rho_b + \rho_f$$

$$\nabla\cdot\vec{D} = \rho_f \;\leftrightarrow\; \oint\vec{D}\cdot d\vec{a}\,' = Q_f$$

$$\vec{P} = \varepsilon_0\chi_e\vec{E}$$

$$\vec{D} = \varepsilon_0(1+\chi_e)\vec{E} = \varepsilon\vec{E}$$

$$\varepsilon_r = 1 + \chi_e = \varepsilon/\varepsilon_0$$

# Physics 435 – Equation Sheet for Midterm 3

## Magnetostatics

$\vec{F} = q\vec{v}\times\vec{B}$ $\quad \vec{\mathscr{r}} = \vec{r} - \vec{r}'$

$\vec{F} = \int dl\, \vec{I}\times\vec{B}$

$$\vec{B}(\vec{r}) = \frac{\mu_0}{4\pi}\int \frac{d\vec{I}\times\hat{\mathscr{r}}}{\mathscr{r}^2}$$

$$\vec{A}(\vec{r}) = \frac{\mu_0}{4\pi}\int \frac{d\vec{I}}{\mathscr{r}}$$

$d\vec{I} \to \vec{J}\, d\tau'$ (volume)
$d\vec{I} \to \vec{k}\, da'$ (area)
$d\vec{I} \to \vec{I}\, dl'$ (line)

$\vec{I} = \lambda\vec{v}$ (line current)
$\vec{k} = \sigma\vec{v}$ (sheet current)
$\vec{J} = \rho\vec{v}$ (volume current)

$\vec{B} = \nabla\times\vec{A}$

$$W = \frac{1}{2\mu_0}\int \left|\vec{B}\right|^2 d\tau$$

## Auxiliary field $\vec{H}$

$$\vec{H} \equiv \frac{1}{\mu_0}\vec{B} - \vec{M}$$

$\nabla\times\vec{H} = \vec{J}_f \quad \leftrightarrow \quad \oint \vec{H}\cdot d\vec{l} = I_f$

$\vec{M} = \chi_m\vec{H}$

$\vec{B} = \mu\vec{H} \qquad \mu = \mu_0(1+\chi_m)$

## B-fields for simple currents

$\vec{B} = \frac{\mu_0 I}{2\pi s}\hat{\phi}$ (infinite line current along *z*-axis)

$\vec{B} = \pm\frac{\mu_0 k}{2}\hat{y}$ (infinite sheet current in *xy* plane, with current in *x*-direction)

$\vec{B} = \mu_0 n I\hat{z}$ (infinite solenoid along *z*-axis)

$\vec{B} = \frac{\mu_0 N I}{2\pi s}\hat{\phi}$ (torus with radius s and symmetry axis along *z*)

## Continuity equation, Ampere's Law for $\vec{A}$

$$\nabla\cdot\vec{J} = -\frac{d\rho}{dt}$$

$\nabla(\nabla\cdot\vec{A}) - \nabla^2\vec{A} = \mu_0\vec{J}$

$\nabla^2\vec{A} = -\mu_0\vec{J}$ (Coulomb gauge)

## Magnetization

$\vec{m} = \mathrm{I}\int d\vec{a}'$

$\vec{\tau} = \vec{m}\times\vec{B}$

$$\vec{M} = \frac{d\vec{m}}{d\tau}$$

$$\vec{A}_{\text{mag}}(\vec{r}) = \frac{\mu_0}{4\pi}\left[\int \frac{\vec{J}_b(\vec{r}')}{\mathscr{r}}\, d\tau' + \oint \frac{\vec{k}_b(\vec{r}')}{\mathscr{r}}\, da'\right]$$

$\vec{J}_b \equiv \nabla'\times\vec{M}$ (bound volume current)

$\vec{k}_b \equiv \vec{M}\times\hat{n}$ (bound sheet current)

# Physics 435 – Equation Sheet for Midterm 3

## Spherical Coordinates

*Line Element:* $d\vec{l} = dr\,\hat{r} + r\,d\theta\,\hat{\theta} + r\sin\theta\,d\phi\,\hat{\phi}$

$$x = r\sin\theta\cos\phi \qquad \hat{x} = \sin\theta\cos\phi\,\hat{r} + \cos\theta\cos\phi\,\hat{\theta} - \sin\phi\,\hat{\phi}$$

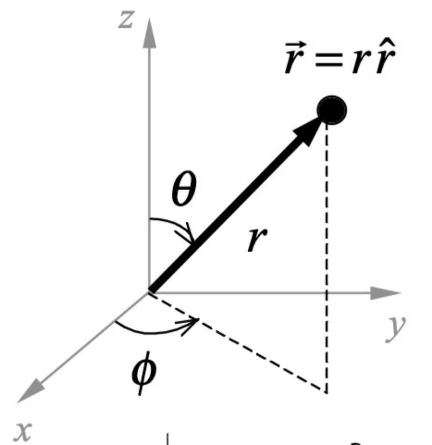
$$y = r\sin\theta\sin\phi \qquad \hat{y} = \sin\theta\sin\phi\,\hat{r} + \cos\theta\sin\phi\,\hat{\theta} + \cos\phi\,\hat{\phi}$$

$$z = r\cos\theta \qquad \hat{z} = \cos\theta\,\hat{r} - \sin\theta\,\hat{\theta}$$

$$r = \sqrt{x^2 + y^2 + z^2} \qquad \hat{r} = \sin\theta\cos\phi\,\hat{x} + \sin\theta\sin\phi\,\hat{y} + \cos\theta\,\hat{z}$$

$$\theta = \tan^{-1}(\sqrt{x^2 + y^2}/z) \qquad \hat{\theta} = \cos\theta\cos\phi\,\hat{x} + \cos\theta\sin\phi\,\hat{y} - \sin\theta\,\hat{z}$$

$$\phi = \tan^{-1}(y/x) \qquad \hat{\phi} = -\sin\phi\,\hat{x} + \cos\phi\,\hat{y}$$




*Gradient*: $$\vec{\nabla}V = \frac{\partial V}{\partial r}\hat{r} + \frac{1}{r}\frac{\partial V}{\partial\theta}\hat{\theta} + \frac{1}{r\sin\theta}\frac{\partial V}{\partial\phi}\hat{\phi}$$

*Laplacian*: $$\nabla^2 V = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial V}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial V}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 V}{\partial\phi^2}$$

*Divergence*: $$\vec{\nabla}\cdot\vec{E} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 E_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\,E_\theta) + \frac{1}{r\sin\theta}\frac{\partial E_\phi}{\partial\phi}$$

*Curl*: $$\vec{\nabla}\times\vec{E} = \frac{\hat{r}}{r\sin\theta}\left[\frac{\partial}{\partial\theta}(\sin\theta\,E_\phi) - \frac{\partial E_\theta}{\partial\phi}\right] + \frac{\hat{\theta}}{r}\left[\frac{1}{\sin\theta}\frac{\partial E_r}{\partial\phi} - \frac{\partial}{\partial r}(rE_\phi)\right] + \frac{\hat{\phi}}{r}\left[\frac{\partial}{\partial r}(rE_\theta) - \frac{\partial E_r}{\partial\theta}\right]$$

| | $\partial_r$ | $\partial_\theta$ | $\partial_\phi$ |
|---|---|---|---|
| $\hat{r}$ | 0 | $\hat{\theta}$ | $\sin\theta\hat{\phi}$ |
| $\hat{\theta}$ | 0 | $-\hat{r}$ | $\cos\theta\hat{\phi}$ |
| $\hat{\phi}$ | 0 | 0 | $-\sin\theta\hat{r}$ $-\cos\theta\hat{\theta}$ |

## Cylindrical Coordinates

*Line Element:* $d\vec{l} = ds\,\hat{s} + s\,d\phi\,\hat{\phi} + dz\,\hat{z}$

$$x = s\cos\phi \qquad \hat{x} = \cos\phi\,\hat{s} - \sin\phi\,\hat{\phi}$$

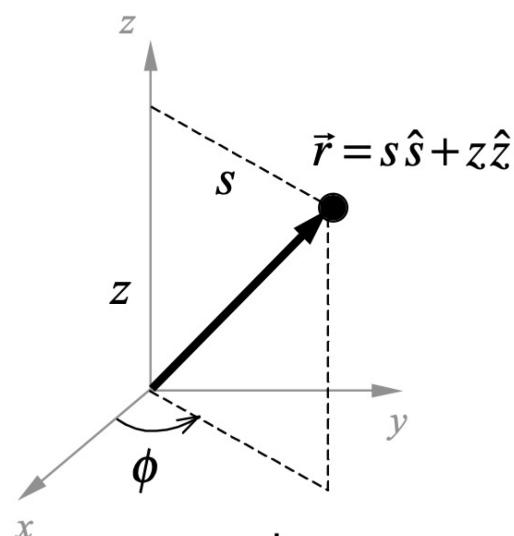
$$y = s\sin\phi \qquad \hat{y} = \sin\phi\,\hat{s} + \cos\phi\,\hat{\phi}$$

$$z = z \qquad \hat{z} = \hat{z}$$

$$s = \sqrt{x^2 + y^2} \qquad \hat{s} = +\cos\phi\,\hat{x} + \sin\phi\,\hat{y}$$

$$\phi = \tan^{-1}(y/x) \qquad \hat{\phi} = -\sin\phi\,\hat{x} + \cos\phi\,\hat{y}$$

$$z = z \qquad \hat{z} = \hat{z}$$




*Gradient*: $$\vec{\nabla}V = \frac{\partial V}{\partial s}\hat{s} + \frac{1}{s}\frac{\partial V}{\partial\phi}\hat{\phi} + \frac{\partial V}{\partial z}\hat{z}$$

*Laplacian*: $$\nabla^2 V = \frac{1}{s}\frac{\partial}{\partial s}\left(s\frac{\partial V}{\partial s}\right) + \frac{1}{s^2}\frac{\partial^2 V}{\partial\phi^2} + \frac{\partial^2 V}{\partial z^2}$$

*Divergence*: $$\vec{\nabla}\cdot\vec{E} = \frac{1}{s}\frac{\partial}{\partial s}(sE_s) + \frac{1}{s}\frac{\partial E_\phi}{\partial\phi} + \frac{\partial E_z}{\partial z}$$

*Curl*: $$\vec{\nabla}\times\vec{E} = \left[\frac{1}{s}\frac{\partial E_z}{\partial\phi} - \frac{\partial E_\phi}{\partial z}\right]\hat{s} + \left[\frac{\partial E_s}{\partial z} - \frac{\partial E_z}{\partial s}\right]\hat{\phi} + \frac{1}{s}\left[\frac{\partial}{\partial s}(sE_\phi) - \frac{\partial E_s}{\partial\phi}\right]\hat{z}$$

| | $\partial_s$ | $\partial_\phi$ | $\partial_z$ |
|---|---|---|---|
| $\hat{s}$ | 0 | $\hat{\phi}$ | 0 |
| $\hat{\phi}$ | 0 | $-\hat{s}$ | 0 |
| $\hat{z}$ | 0 | 0 | 0 |