Midterm Exam 2 Formula Sheet

April. 4, 2023

Reference formulae

Time-dependent Schrödinger equation: $i\hbar\frac{\partial}{\partial t}\psi(x,t) = -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}\psi(x,t) + V(x,t)\psi(x,t)$

Normalization: $\int_{-\infty}^{\infty} dx\, |\psi(x,t)|^2 = 1$

Expectation values: $\langle \mathcal{O} \rangle = \int_{-\infty}^{\infty} dx\, \psi^*(x,t)\mathcal{O}\psi(x,t)$

standard deviation σ: $\sigma_{\mathcal{O}}^2 = \langle \mathcal{O}^2 \rangle - \langle \mathcal{O} \rangle^2$

Time-independent Schrödinger equation: $H\psi_n(x) = -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}\psi_n(x) + V(x)\psi_n(x) = E_n\psi_n(x)$

$\psi(x,t) = \sum_n c_n\psi_n(x)e^{-iE_nt/\hbar}$

Orthonormality: $\int_{-\infty}^{\infty} dx\, \psi_m^*(x)\psi_n(x) = \delta_{mn}$

Operators: momentum $p \leftrightarrow -i\hbar\frac{\partial}{\partial x}$; position $x \leftrightarrow x$; Hamiltonian $H \leftrightarrow p^2/2m + V(x,t)$

Commutator $[A,B] = AB - BA$; $[x,p] = i\hbar$

Uncertainty $\sigma_A^2\sigma_B^2 \geq \left(\frac{1}{2i}\langle [A,B] \rangle\right)^2$

$\frac{d}{dt}\langle Q \rangle = (i/\hbar)\langle [H,Q] \rangle + \langle \partial Q/\partial t \rangle$

Infinite square well, $V(x) = 0$ for $0 < x < L$, $V(x) = \infty$ elsewhere:

$\psi_n(x) = \sqrt{\frac{2}{L}}\sin(n\pi x/L)$, $E_n = \frac{\hbar^2}{2m}\left(\frac{\pi n}{L}\right)^2$

Free particle, $V = 0$. Momentum eigenstates $\psi_k(x) = \frac{1}{\sqrt{2\pi}}e^{ikx}$

Continuum orthonormality, $\int_{-\infty}^{\infty} dx\, \psi_{k_1}^*(x)\psi_{k_2}(x) = \delta(k_1 - k_2)$

Integrals: Gaussian, $\int_{-\infty}^{\infty} dx\, e^{-(\alpha x^2 + \beta x)} = \sqrt{\pi/\alpha}\, e^{\beta^2/4\alpha}$ for $\mathrm{Re}\,\alpha > 0$

$\int_{-\infty}^{\infty} dx\, x^2 e^{-\alpha x^2} = \sqrt{\pi/4\alpha^3}$

Delta function, $\delta(x) = 0$ for $x \neq 0$, ∞ for $x = 0$, $\int_{-\infty}^{\infty} dx\, \delta(x-a)f(x) = f(a)$

Fourier transform: $f(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} dk \tilde{f}(k)e^{ikx}$, $\tilde{f}(k) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} dx f(x)e^{-ikx}$

Simple harmonic oscillator, $V(x) = \frac{1}{2}m\omega^2x^2$: define $x_0^2 \equiv \hbar/m\omega$, then

$\psi_n(x) = A_n e^{-x^2/2x_0^2}H_n(x/x_0)$, $A_n = (2^n n! x_0\sqrt{\pi})^{-1/2}$, Hermite polynomials $H_n(y)$: $H_0(y) = 1$

$H_1(y) = 2y$, $H_2(y) = 4y^2 - 2$, $H_3(y) = 8y^3 - 12y$, $H_4(y) = 16y^4 - 48y^2 + 12$

$E_n = (n + 1/2)\hbar\omega$

Raising and lowering operators,

$$a = \frac{1}{\sqrt{2}}\left(\frac{x}{x_0} + i\frac{x_0 p}{\hbar}\right), \qquad a^\dagger = \frac{1}{\sqrt{2}}\left(\frac{x}{x_0} - i\frac{x_0 p}{\hbar}\right), \qquad [a, a^\dagger] = 1$$

Laplacian in spherical coordinates:

$$\nabla^2 = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial\phi^2}$$

Angular momentum operators
Commutation relations: $[L_x, L_y] = i\hbar L_z$, $[L_y, L_z] = i\hbar L_x$, $[L_z, L_x] = i\hbar L_y$, $[L_z, L^2] = 0$
In spherical coordinates:

$$L^2 = -\hbar^2\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\right] \qquad L_z = -i\hbar\frac{\partial}{\partial\phi}$$

Raising and lowering operators,

$$L_\pm = L_x \pm iL_y = \hbar e^{\pm i\phi}\left(i\cot\theta\frac{\partial}{\partial\phi} \pm \frac{\partial}{\partial\theta}\right), \qquad [L_z, L_\pm] = \pm\hbar L_\pm$$

First few spherical harmonics:

$$Y_{00} = \sqrt{\frac{1}{4\pi}} \qquad Y_{1\pm1} = \mp\sqrt{\frac{3}{8\pi}}\sin\theta e^{\pm i\phi} \qquad Y_{10} = \sqrt{\frac{3}{4\pi}}\cos\theta$$

$$Y_{2\pm2} = \sqrt{\frac{15}{32\pi}}\sin^2\theta e^{\pm 2i\phi} \qquad Y_{2\pm1} = \mp\sqrt{\frac{15}{8\pi}}\sin\theta\cos\theta e^{\pm i\phi} \qquad Y_{20} = \sqrt{\frac{5}{16\pi}}(3\cos^2\theta - 1)$$