P524: Survey of Instrumentation and Laboratory Techniques
Week 11

11/4/2025

Error analysis and reporting (today)
Cryogenics (Thursday)

Errors: Definitions

Results of a measurement will usually have:

Accuracy: How close the result is to a "true" value

Precision: how well the result is determined and/or reproducible

"True" value can include:

- Theoretically exact value
- Accepted value based on unfalsified previous measurements

} Difference between these and result: "discrepancy" or simply "comparison:" $\frac{measured - theory}{theory}$

- Usually: The actual, unbiased best estimate sought in the measurement (see next slides…)

Accuracy generally depends on *systematic* errors

Mis-calibration (instrument reads high or low)

Bias (my readings skew high or low)

Assessed by measuring the same quantity with different instruments or methods (cannot improve with averaging)

NOTE: Difference from an accepted value may be a systematic error (e.g., a thermometer offset) OR a *mistake* (e.g., we measured the specific heat of a different alloy of aluminum than the reference).

Precision generally depends on *statistical* or *random* errors or "uncertainties"

Presumes multiple measurements; can improve with averaging

Statistical Errors: Mean and Standard Deviation

If there are random errors, multiple measurements x_i will be described by a probability distribution *P(x)*

Want: Maximum value of the distribution P_{max}

Good, practical estimate of P_{max}: mean: $$\mu = \frac{1}{N}\sum_{i=1}^{N} x_i$$

Good, practical estimate of the random error:

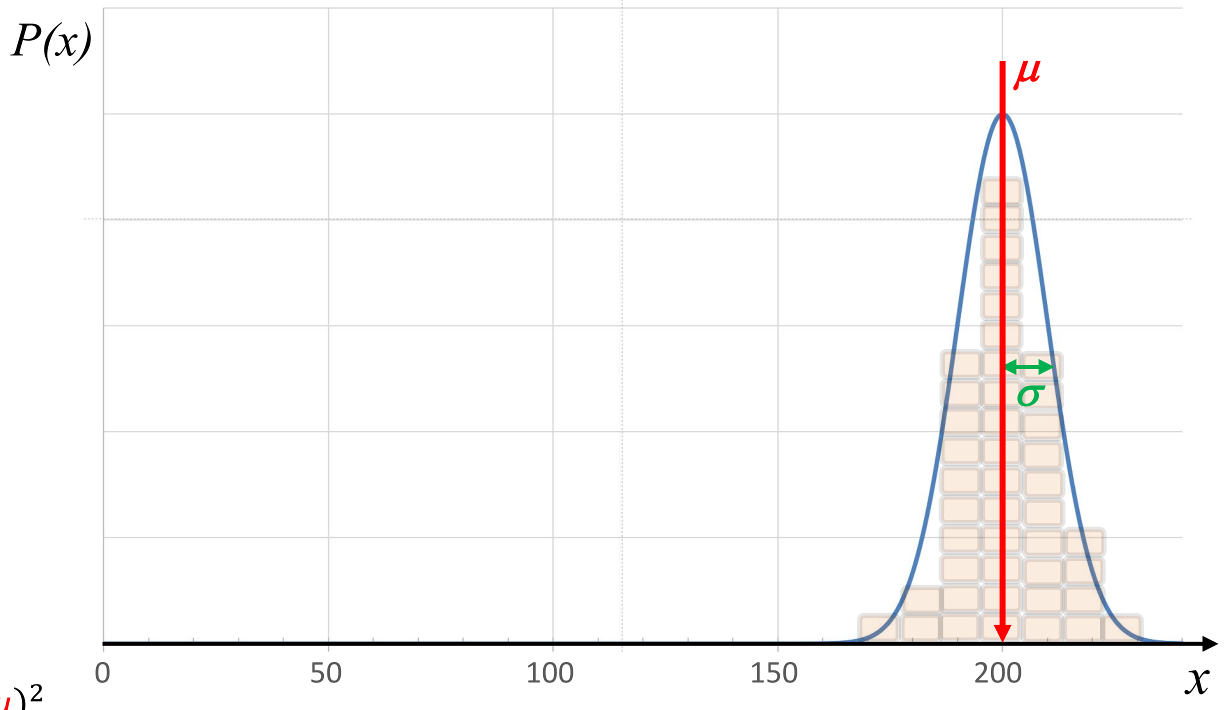
deviation: difference between any one measurement and the mean: $(x_i - \mu)$

Average deviation: 0 (+ and – values equally probable)

So define *variance*: mean square deviation: $$\sigma^2 = \frac{1}{N}\sum_{i=1}^{N}(x_i - \mu)^2$$

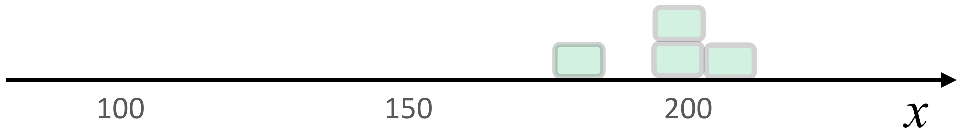
standard deviation: root mean square deviation: $$\sigma = \sqrt{\frac{1}{N}\sum_{i=1}^{N}(x_i - \mu)^2}$$

In practice: sample standard deviation: $$s = \sqrt{\frac{1}{(N-1)}\sum_{i=1}^{N}(x_i - \mu)^2}$$



Example: Specific heat of antimony

Trial	Value [J/(kg K)]
1	202
2	212
3	179
4	200



Mean: μ = (202+212+179+200)/4 = 198.25

Standard Deviation: $s = \sqrt{\frac{1}{(N-1)}\sum_{i=1}^{N}(x_i - \mu)^2}$ = 13.86524

N < 5 estimate: σ =(212-179)/2 = 16.5 (half the range of all of the data)

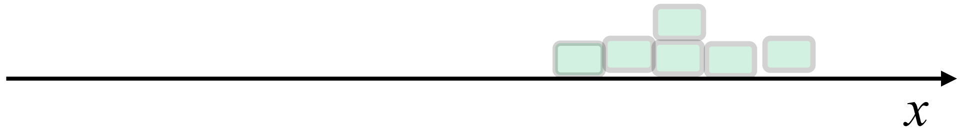
Best estimate of the true value is 198.25 ± 13.86524 ?

Standard Error: $\sigma_\mu = \sigma/\sqrt{N} = 13.86524/\sqrt{4}$ = 6.93262

Best estimate of the true value is 198 ± 7 or 198.3 ± 6.9

Example: Specific heat of antimony

Trial	Value[J/(kg K)]
1	202
2	212
3	179
4	200
5	184
6	222



Mean: μ = (202+212+179+200+184+221)/6 = 199.8333

Standard Deviation: $s = \sqrt{\frac{1}{(N-1)}\sum_{i=1}^{N}(x_i-\mu)^2}$ = 16.30235

N > 5 estimate: σ = (212-184)/2 = 14 (half the range of the 2/3 of the measurements closest to mean)

Standard Error: $\sigma_\mu = \sigma/\sqrt{N} = 16.30235/\sqrt{6}$ = 6.65540

So the best estimate of the true value is 199.8 ± 6.7

Significant Digits

- Few quantities are known with unlimited precision. Reported measurements should keep only those digits which actually contribute to the known resolution. These are "significant digits" or "significant figures" (s.f.).

  (My weight on a scale that measures to the nearest pound = 150 lb. It is false to report 150.0 lb.)

- Rules for s.f.
  - All non-zero digits are significant. [23, 2.3, 2.34]
  - Zeros between non-zero digits are significant. [203, 2.304]
  - In a number with a decimal point, trailing zeros (to the right of the last non-zero digit) are significant. [2.30, 2.340]
  - Leading zeros are not significant (they are placeholders indicating the scale of a number). [023, 0.23, 0.023]
  - In a number without a decimal point, trailing zeros are ambiguous (may be significant or placeholders) [230, 2300]; *use scientific notation to resolve this*

Significant Digits

Number	significant digits	
2.34	3	
2.340	4	
2.3	2	
234	3	
0.00234	3	
230	2 or 3	*Ambiguous because we use “normal” notation*
2.3×10^2	2	*Scientific notation resolves the ambiguity*
2.30×10^2	3	
2.34×10^2	3	

Significant Digits – Arithmetic

- Rule for **multiplication and division**: Final output should have the same number of s.f. as the (measured) input with the least s.f.

Suppose I measure 2 lengths: 10.000 cm and 6.0 cm

then 10.000/6.0 = 1.7

$1.2\ \text{cm} \times 1.2\ \text{cm} = 1.4\ \text{cm}^2$

- Rule for **addition and subtraction**: Last significant *decimal place* of the final output should match that of the measured input with least resolution.

10.000 cm + 6.0 cm = 16.0 cm

Significant digits and uncertainty

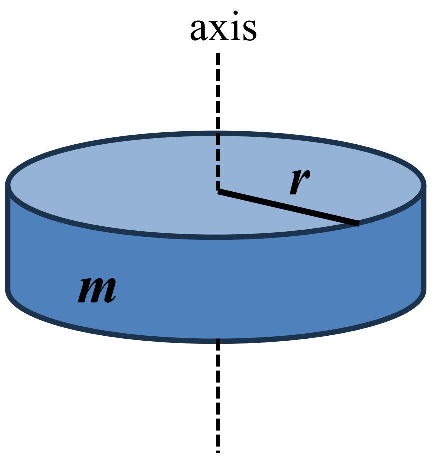
- When doing calculations with data, keep as many digits as you can until you reach the end
- When reporting a final result, give uncertainty with two significant figures, then round off your answer to the same digit as the second digit in your uncertainty:

$$m = 35.8\underline{21}34 \pm 0.0\underline{61}352\text{kg}$$

Match decimal from δμ   Keep two digits

$$m = 35.821 \pm 0.061\text{ kg}$$

Propagation of Uncertainty



Solid disk of mass m, radius r :

Moment of inertia, I :

$$I = \frac{1}{2} m r^2$$

Taylor expansion:

$$I = I_0 + \frac{\partial I}{\partial m}(m - m_0) + \frac{\partial I}{\partial r}(r - r_0) + \frac{1}{2!}\frac{\partial^2 I}{\partial m^2}(m - m_0)^2 + \cdots$$

$take\ I_0 = \bar{I}, m_0 = \bar{m}, etc.$

$$I - \bar{I} = \frac{\partial I}{\partial m}(m - \bar{m}) + \frac{\partial I}{\partial r}(r - \bar{r})$$

Variance of I :

$$\sigma_I^2 = \frac{1}{N}\sum_{i=1}^{N}(I_i - \bar{I})^2$$

Substitute :

$$\sigma_I^2 = \frac{1}{N}\sum_{i=1}^{N}\left(\frac{\partial I}{\partial m}(m - \bar{m}) + \frac{\partial I}{\partial r}(r - \bar{r})\right)^2$$

$$\sigma_I^2 = \frac{1}{N}\sum_{i=1}^{N}\left[\left(\frac{\partial I}{\partial m}\right)^2(m - \bar{m})^2 + \left(\frac{\partial I}{\partial r}\right)^2(r - \bar{r})^2 + 2\left(\frac{\partial I}{\partial m}\right)\left(\frac{\partial I}{\partial r}\right)(m - \bar{m})(r - \bar{r})\right]$$

$$\sigma_I^2 = \left(\frac{\partial I}{\partial m}\right)^2\frac{1}{N}\sum_{i=1}^{N}(m - \bar{m})^2 + \left(\frac{\partial I}{\partial r}\right)^2\frac{1}{N}\sum_{i=1}^{N}(r - \bar{r})^2 + 2\left(\frac{\partial I}{\partial m}\right)\left(\frac{\partial I}{\partial r}\right)\frac{1}{N}\sum_{i=1}^{N}(m - \bar{m})(r - \bar{r})$$

$$\sigma_I^2 = \left(\frac{\partial I}{\partial m}\right)^2\sigma_m^2 + \left(\frac{\partial I}{\partial r}\right)^2\sigma_r^2 + 2\left(\frac{\partial I}{\partial m}\right)\left(\frac{\partial I}{\partial r}\right)\sigma_{mr}^2$$

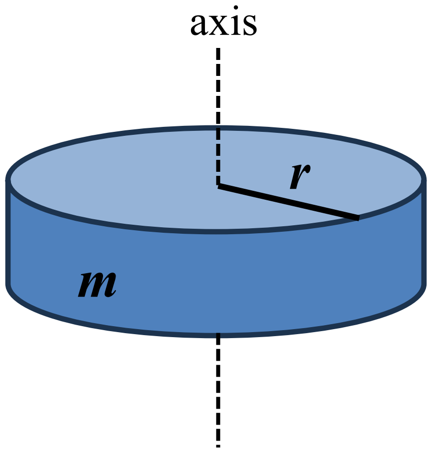
$$\sigma_{mr}^2 = \frac{1}{N}\sum_{i=1}^{N}(m - \bar{m})(r - \bar{r}) = \text{"}covariance\text{"}$$

small if m, r uncorrelated

$$\sigma_I \approx \sqrt{\left(\frac{\partial I}{\partial m}\right)^2\sigma_m^2 + \left(\frac{\partial I}{\partial r}\right)^2\sigma_r^2}$$

Error propagation equation

Propagation of Uncertainty



Solid disk of mass $m \pm \delta m$, radius $r \pm \delta r$

Calculate $I \pm \delta I$

$$I = \frac{1}{2} m r^2$$

$$\delta I_m = \frac{\partial I}{\partial m} \delta m$$

$$\delta I_m = \left(\frac{1}{2} r^2\right) \delta m$$

$$\delta I_r = \frac{\partial I}{\partial r} \delta r$$

$$\delta I_r = \left(\frac{1}{2} m(2r)\right) \delta r = m r \delta r$$

$$\delta I_{total} = \sqrt{{\delta I_m}^2 + {\delta I_r}^2}$$

Simplify (in terms of fractional errors):

$$\frac{\delta I_m}{I} = \frac{\left(\frac{1}{2} r^2\right) \delta m}{\left(\frac{1}{2} m r^2\right)} = \frac{\delta m}{m} \Rightarrow \delta I_m = I \frac{\delta m}{m}$$

$$\frac{\delta I_r}{I} = \frac{m r \delta r}{\left(\frac{1}{2} m r^2\right)} = \frac{2 \delta r}{r} \Rightarrow \delta I_r = 2I \frac{\delta r}{r}$$

$$\delta I_{total} = \sqrt{{\delta I_m}^2 + {\delta I_r}^2}$$

$$= \sqrt{\left(I \frac{\delta m}{m}\right)^2 + \left(2I \frac{\delta r}{r}\right)^2}$$

$$\delta I_{total} = I \sqrt{\left(\frac{\delta m}{m}\right)^2 + \left(2 \frac{\delta r}{r}\right)^2}$$

Propagation of Uncertainty: on the Mean

general mean: $\mu = \frac{1}{N}\sum_{i=1}^{N} x_i$

Error propagation: $\sigma_\mu \approx \sqrt{\left(\frac{\partial\mu}{\partial x_1}\right)^2 \sigma_{x_1}^2 + \left(\frac{\partial\mu}{\partial x_2}\right)^2 \sigma_{x_2}^2 + \cdots + \left(\frac{\partial\mu}{\partial x_N}\right)^2 \sigma_{x_N}^2}$

$$\frac{\partial\mu}{\partial x_1} = \frac{1}{N}\frac{\partial}{\partial x_1}(x_1 + x_2 + \cdots + x_N)$$

$$= \frac{1}{N}\ (1 + 0 + \cdots + 0) = \frac{1}{N}$$

$$\frac{\partial\mu}{\partial x_2} = \cdots = \frac{\partial\mu}{\partial x_N} = \frac{1}{N}$$

$$\sigma_\mu \approx \sqrt{\left(\frac{1}{N}\right)^2 \sigma_{x_1}^2 + \left(\frac{1}{N}\right)^2 \sigma_{x_2}^2 + \cdots + \left(\frac{1}{N}\right)^2 \sigma_{x_N}^2}$$

$$\sigma_\mu \approx \left(\frac{1}{N}\right)\sqrt{\sigma_{x_1}^2 + \sigma_{x_2}^2 + \cdots + \sigma_{x_N}^2}$$

$$assume\ \sigma_{x_1} \approx \sigma_{x_2} \approx \cdots \approx \sigma_{x_N} = \sigma$$

$$\sigma_\mu \approx \left(\frac{1}{N}\right)\sqrt{N\sigma^2} = \frac{\sigma}{\sqrt{N}}$$

Summary

Take multiple measurements

Calculate the mean and standard deviation of the mean

For derived quantities, use the error propagation equation

Report results as mean +/- standard deviation of the mean: $\mu \pm \sigma_{\mu}$

Round the error (std. dev. of mean) to 1 or 2 significant digits, then round the mean to same precision as error

If another method or instrument can be used and yields a different result, report the difference σ_{sys} as: $\mu \pm \sigma_{\mu(stat)} \pm \sigma_{sys}$

Label comparisons with theory or accepted values as such; compute as fractional differences: $\frac{\mu_{meas} - \mu_{theory}}{\mu_{theory}}$