Chapter 7

Weak Interaction

1. Historical Overview

Radioactivity was discovered by Becquerel in 1896 and β-ray was soon identified as electrons. In 1914 Chadwick observed that the energy of electrons emitted in β-decay is continuous. This puzzling discovery suggested violation of energy and angular momentum conservation. In the early 1930s, Pauli proposed the existence of a weakly interacting fermion, neutrino, as a solution to this puzzle. Soon afterwards, Fermi put forward his theory of nuclear β-decay in 1933.

Although the study of weak interaction was once limited to nuclear β decays, we now know that weak interaction is present for all quarks and leptons, although the weak interaction is often masked by the much stronger electromagnetic and strong interactions.

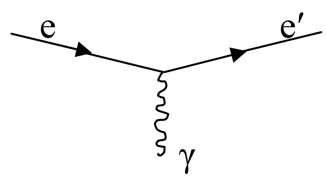
The Lagrangian density (L) for electromagnetic interaction can be written as

$$L_{Int}^{EM} = -e\bar{\psi}\gamma_\mu \psi A^\mu = -J_\mu A^\mu \qquad (7.1)$$

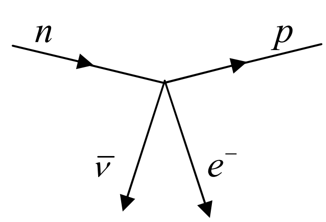
where

$$J_\mu = e\bar{\psi}\gamma_\mu\psi$$

graphically,



Fermi assumed that nucleon β-decay is represented by a similar diagram where the EM field is replaced by the $\bar{\nu}e^-$ current:



Fermi suggested the following substitutions to go from EM interaction to weak interaction:

$$\begin{aligned} \bar{\psi}_e \gamma_\mu \psi_e &\to \bar{\psi}_p \gamma_\mu \psi_n \\ A^\mu &\to \bar{\psi}_e \gamma^\mu \psi_\nu \\ e &\to G_F \big/ \sqrt{2} \end{aligned} \tag{7.2}$$

The Fermi coupling constant, G_F, was to be determined by experiment and was found to be

$$G_F = 1.03 \times 10^{-5} \left(m_p\right)^{-2} \tag{7.3}$$

The Lagrangian density for β-decay was therefore given as

$$L_\beta = -\frac{G_F}{\sqrt{2}} \bar{\psi}_p \gamma_\mu \psi_n \bar{\psi}_e \gamma^\mu \psi_{\nu e} \tag{7.4}$$

In 1934 β^+ decay was observed by Curie and Joliot. Later, Alvarez observed electron capture in 1938. Since Equation 7.4 only describes an emission of e- (or an absorption of e+), in order to describe β^+ decay and electron capture the Lagrangian density needs to be generalized to

$$L_\beta = -\frac{G_F}{\sqrt{2}} \left[\bar{\psi}_p \gamma_\mu \psi_n \bar{\psi}_e \gamma^\mu \psi_{\nu e} + \bar{\psi}_n \gamma_\mu \psi_p \bar{\psi}_{\nu e} \gamma^\mu \psi_e \right] \tag{7.5}$$

In other words, the Hermitian conjugate of Equation 7.4 is added to the Lagrangian density.

Note that L_β is a sum of scalar product of two Lorentz 4-vectors, and is invariant with respect to Lorentz transformation and spatial inversion.

In 1936 Gamow and Teller pointed out that L_β can contain other terms too without violating parity and L_β was generalized to

$$L_\beta = \sum_j \left(-\frac{G_F}{\sqrt{2}} \right) \left(\bar{\psi}_p O_j \psi_n \bar{\psi}_e O_j \bar{\psi}_{\nu e} + h.c. \right) \tag{7.6}$$

where

$$O_j = 1,\ \gamma^5,\ \gamma_\mu,\ \sigma_{\mu\nu},\ \gamma_\mu\gamma^5 \tag{7.7}$$

representing scalar, pseudoscalar, vector, tensor, axial-vector (*S*, *P*, *V*, *T*, *A*) interaction respectively.

For nuclear β-decays, momentum and energy transfers are very small, and one can use non-relativistic wave function for the nucleons:

$$\psi = \begin{pmatrix} \phi \\ 0 \end{pmatrix} \tag{7.8}$$

where the relativistic 'small' component is set to zero.

Note that
$$\bar{\psi}_p 0_j \psi_n = \left(\phi_p^*, 0\right) \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} 0_j \begin{pmatrix} \phi_n \\ 0 \end{pmatrix} \tag{7.9}$$

$0_j = 1,\ \gamma_\mu,\ \sigma_{\mu\nu},\ \gamma_\mu\gamma^5$ all have non-vanishing diagonal elements. In contrast, $\gamma^5 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ has only off-diagonal elements. Hence,

$$\bar{\psi}_p \gamma^5 \psi_n = 0 \tag{7.10}$$

and the pseudoscalar term can be ignored in nuclear β-decay. From Equation 7.8, we obtain for the various couplings

$$\begin{aligned}
&S: \quad \bar{\psi}_p \psi_n = \phi_p^+ \phi_n \\
&V: \quad \bar{\psi}_p \gamma_\mu \psi_n = \phi_p^+ \phi_n \ \text{ for } u = 0; \ = 0 \text{ for } \mu = 1,\ 2,\ 3 \\
&T: \ \bar{\psi}_p \sigma^{\mu\nu} \psi_n = \phi_p^+ \sigma^k \phi_n \ \text{ for } \mu,\nu = 1,\ 2,\ 3 \text{ cyclic}; \ = 0 \text{ otherwise} \\
&A: \bar{\psi}_p \gamma^5 \gamma^\mu \psi_n = -\phi_p^+ \sigma^k \phi_n \ \text{ for } \mu = k = 1,2,3; \ = 0 \text{ for } \mu = 0
\end{aligned} \tag{7.11}$$

From Equation 7.11 we conclude that the scalar and vector couplings cannot induce spin-flip transitions (since spin-up ϕ_n can not couple to spin-down ϕ_p, for example). The axial-vector and tensor couplings can cause spin-flip transition (since σ^k contains non-zero off-diagonal matrix elements, allowing coupling between spin-up ϕ_n to couple to spin-down ϕ_p, for example). Therefore, we have

$$\begin{aligned} &S,V \text{ couplings}: \Delta J = 0 \quad \text{(no spin-flip)} \\ &A,T \text{ couplings}: \Delta J = 0, \ \pm 1 \quad (\text{except } J = 0 \not\Rightarrow J = 0) \end{aligned} \tag{7.12}$$

Experimentally, $|\Delta J| = 1$ spin-flip transitions, as well as $J = 0 \Rightarrow J = 0$ transitions were observed in weak decays. Therefore the axial and/or tensor terms in $\mathcal{L}_\beta$ has to be non-zero. Similarly, *S* and/or *V* coupling has to be non-zero. The determination of the magnitudes of C_j had to wait for two decades.

The discovery of muons in the 1930s and their decays as well as the pions, kaons, and hyperons and their decays, suggested that they have similar characteristics and coupling strength. The nuclear β-decay phenomenon was therefore generalized to cover other weak-decay processes. The idea of 'universal charged weak interaction' was put forward to describe various decay processes as simply different manifestation of a general Fermi interaction.

In 1956, the τ/θ puzzle, where

$$\theta \to \pi^+\pi^0 \text{ (even parity)}$$
$$\tau \to \pi^+\pi^+\pi^- \text{ (odd parity)}$$

were observed for two particles, θ and τ, of similar if not identical masses. This prompted Lee and Yang to suggest that parity could be violated in weak decay. In fact, they pointed out that parity conservation was never tested in β-decay experiments.

Lee and Yang's suggestion that parity might not be conserved in weak interaction was soon confirmed by Wu et al., who used polarized $^{60}Co(5^+)$ and found that the e^- from $^{60}\text{Co} \to ^{60} N_i^*\left(4^+\right) + e^- + \bar{\nu}_e$ decay were emitted preferentially opposite to the spin orientation of ^{60}Co. This implies that the $\vec{S}\bullet\vec{P}$ term, which is odd under space inversion, is non-vanishing, hence parity is violated.

Subsequent experiments by Garwin, Lederman, Weinrich and by Friedman and Telegdi confirmed that parity was violated in the
$\pi^+ \to \mu^+ + \nu_\mu$ and $\mu^+ \to e^+ + \nu_e + \bar{\nu}_\mu$ decays.

The observation of parity violation in weak interaction implies that L_β can have both a scalar component and a pseudoscalar component:

Scalar

$$L_\beta = -\frac{G_F}{\sqrt{2}} \sum_j \left[\overline{\psi}_p O_j \psi_n\right]\left[\overline{\psi}_e O_j \left(C_j - C'_j \gamma_5\right)\psi_{\nu e}\right] \tag{7.13}$$

pseudoscalar

The coefficients C_j and C'_j have to be determined by experiments. They can be complex numbers if time-reversal invariance is violated. Hence, a total of 10 coupling constants (C_j and C'_j for S, P, V, A, T) with 19 real constants need to be determined (eliminating one arbitrary phase)!

Frauenfelder et al. found that electrons emitted in ^{60}Co β-decay are longitudinally polarized with helicity consistent with $-v/c$. Similarly, positrons were found to have a helicity of v/c. These results showed that weak interactions result in electrons which are left-handed and positrons of right-handedness in relativistic limit $v \rightarrow c$.

The observed handedness of electrons and positrons implies that

$$\begin{aligned} C'_j &= +C_j \text{ for } V \text{ or } A \text{ coupling} \\ C'_j &= -C_j \text{ for } S,\ P,\ T \text{ couplings} \end{aligned} \tag{7.14}$$

Eq. 7.14 can be understood by realizing that for S coupling, we have $\overline{\psi}_e \psi_{\nu e} = (\overline{\psi}_e)_L(\psi_{\nu e})_R + (\overline{\psi}_e)_R(\psi_{\nu e})_L$. Therefore, a left-handed electron can only couple to a right-handed neutrino for S coupling. Since $(\psi_{\nu e})_R = 1/2(1+\gamma_5)\psi_{\nu e}$, we conclude from Eq. 7.13 that $C_S{}' = -C_S$. Other results in Eq. 7.14 can be obtained with similar considerations.

In order to distinguish the two possible scenarios in Equation 7.14, it was necessary to determine the ν_e helicity. This was accomplished in an ingenious experiment by Goldhaber, Grodzins, and Sunyar who used the following reactions:

$$e^- + {}^{152}Eu \rightarrow \nu_e + {}^{152}Sm^*\left(1^-\right) \qquad (7.15)$$

$$(0-) \qquad\qquad \hookrightarrow {}^{152}Sm\ (0^+) + \gamma$$

From the measurement of the circular polarization of γ, it was concluded that ν_e has a negative helicity: $h(\nu_e) = -1$. This result showed that S, P, T coupling sin Equation 7.14 are small if not zero. Therefore, only C_A and C_V are mainly responsible for β-decays.

From the rates of the ${}^{10}C \rightarrow {}^{10}B$, ${}^{14}O \rightarrow {}^{14}N$, $0^+ \rightarrow 0^+$ transition, where only C_V can contribute, one determined that

$$C_V = 1 \qquad (7.16)$$

From neutron lifetime, which is proportional to $C_V^2 + 3C_A^2$, one obtained

$$\left|C_A / C_V\right| = 1.25 \qquad (7.17)$$

The relative phase between C_A and C_V was determined through angular correlation experiment in polarized neutron decay:

$$C_A = +1.25\ C_V \qquad (7.18)$$

Collecting Equations 7.13, 7.14, 7.16, and 7.18, one finds

$$\mathsf{L}_\beta = -\frac{G_F}{\sqrt{2}}\bar{\psi}_p\gamma_\mu\left(1-1.25\gamma_5\right)\psi_n\bar{\psi}_e\gamma^\mu\left(1-\gamma_5\right)\psi_{\nu e} \qquad (7.19)$$

In 1958 Feynman-Gellmann and Sudarshan-Marshak proposed the universal *V-A* form for charged weak current:

$$\mathsf{L}_\beta = -\frac{G_F}{\sqrt{2}}\left(J_\mu J^{\mu+} + J_\mu^+ J^\mu\right) \qquad (7.20)$$

where J_μ is the 'charge-lowering' and J_μ^+ the 'charge-raising' weak current. J_μ consists of both a lepton part and a hadron part

$$J_\mu = J_\mu^{hadron} + J_\mu^{lepton} \qquad (7.21)$$

$$J_\mu^{lepton} = \bar{\psi}_e\gamma_\mu\left(1-\gamma_5\right)\psi_{\nu e} + \bar{\psi}_\mu\gamma_\mu\left(1-\gamma_5\right)\psi_{\nu\mu} + \ldots \qquad (7.22)$$

$$J_\mu^{hadron} = \bar{D}\gamma_\mu\left(1-\gamma_5\right)U \tag{7.23}$$

Note that the deviation of C_A/C_V from 1 (Equation 7.18) was attributed to strong interaction effect in the nucleon.

Equation 7.23 only explained weak decays with $\Delta S = 0$. However, $\Delta S = 1$ weak decays, such as

$$K^+ \to \mu^+\nu,\ \Lambda \to P\pi^-,\ \Sigma^- \to ne^-\nu...$$

had also been observed with reduced strength. In 1963 Cabbibo combined the $\Delta S = 0$, $\Delta S = 1$ transitions by suggesting the following form for the hadronic weak current:

$$J_\mu^{hadron} = \bar{D}_C\gamma_\mu\left(1-\gamma_5\right)U \tag{7.24}$$

where

$$D_C = \cos\theta_C D + \sin\theta_C S \tag{7.25}$$

The weak interaction eigenstate is therefore a mixture of the strange and down quarks. The mixing angle θ_C is determined from the relative strength of the $\Delta S = 0$ versus $\Delta S = 1$ transitions.

$$\theta_C \simeq 13^o \tag{7.26}$$

The Feynman-Gellmann-Cabbibo's scheme is a generalization of Fermi's theory. It suffers the same difficulties of the Fermi's theory, as recognized by Heisenberg in 1936. Namely, the 'contact' interaction violates unitarity at sufficiently high energies. This problem can be illustrated by considering the $\nu_e + e^- \to e^- + \nu_e$ reaction. As will be shown later, the cross-section for this reaction is

$$\sigma = \frac{G_F^2 S}{\pi} \tag{7.27}$$

However, the cross-section can also be expressed as

$$\frac{d\sigma}{d\Omega} = \left|f\left(\theta\right)\right|^2 \tag{7.28}$$

where

$$f(\theta)=\frac{1}{2E}\sum_{\ell=0}^{\infty}(2\ell+1)M_L P_L(\cos\theta)$$

For a contact interaction of zero range, only the S-wave contributes, and Equation 7.28 becomes

$$\frac{d\sigma}{d\Omega}=\frac{1}{4E^2}|M_0|^2=\frac{|M_0|^2}{S} \qquad \sigma=\frac{4\pi}{S}|M_0|^2 \tag{7.29}$$

unitarity implies

$$|M_0|\leq 1 \tag{7.30}$$

Hence

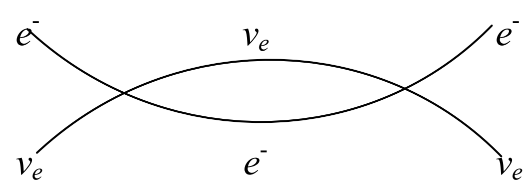
$$\sigma\leq\frac{4\pi}{S} \tag{7.31}$$

From Equations 7.27 and 7.31, one concludes that unitarity limit is violated at

$$\frac{4\pi}{S}=\frac{G_F^2 S}{\pi} \tag{7.32}$$

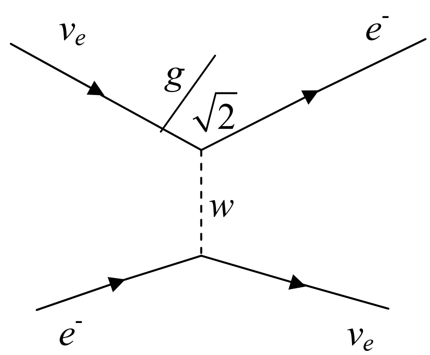
which occurs at $\sqrt{S}\simeq 600\ GeV$

Can second-order diagram cancel the leading-order and make the result finite? It turns out that the second-order diagram for point-interaction such as



actually diverges. In QED, one also encounters divergences. However, they can be removed to all orders by mass and charge renormalization. This is not possible for the Fermi theory.

Another way out is to introduce a vector boson mediating the weak interaction. The $\nu_e\, e^- \rightarrow e^-\, \nu_e$ interaction is therefore



where

$$M = \frac{-ig^2}{2}\left[\bar{u}_e\gamma_\mu \frac{1-\gamma^5}{2}u_{\nu e}\right]\left[\frac{-g^{\mu\nu}+q^\mu q^\nu/\mu_w^2}{q^2-M_w^2}\right]\left[\bar{u}_{\nu e}\gamma_\nu \frac{1-\gamma^5}{2}u_{e^-}\right] \quad (7.33)$$

At low energies, the Fermi theory should emerge. Therefore

$$\frac{G_F}{\sqrt{2}} = \frac{g^2}{8M_w^2} \quad (7.34)$$

The expression of the cross-section resulting from the lowest order diagram is

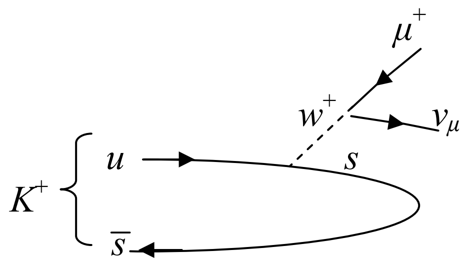
$$\frac{d\sigma}{dy}\left(\nu_e e^- \to e^- \nu_e\right) = \frac{g^4}{32\pi}\frac{S}{\left(q^2-M_w^2\right)^2} \quad (7.35)$$

which does not have the unitarity problem. However, higher order diagrams still diverge. Furthermore, for processes such as

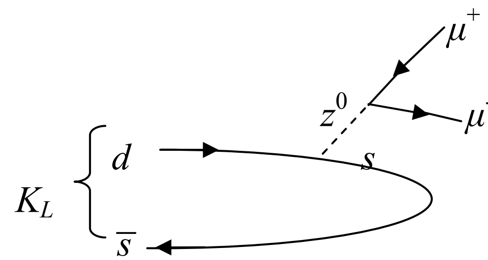
$$\nu_e + \bar{\nu}_e \to w^+ + w^-$$

it can be shown that the cross-section diverges even in the Born term. The origin of this divergence is the $q^\mu q^\nu$ term in the w-propagator, signifying a longitudinally polarized massive w (This term is absent for QED).

Despite the problem encountered by the Intermediate Vector Boson model, it was proposed in the 1960s that there should be neutral weak interaction mediated by neutral vector bosons. However, no neutral weak interaction was observed. In particular, it was very puzzling why $K^+ \to \mu^+\nu_\mu$ has a B.R. of 63%, while $K_L \to \mu^+\mu^-$ has a B.R. of only 9×10^{-9}, which can be accounted for by radiative effect.



$K^+ \to \mu^+ \nu_\mu$



$K_L \to \mu^+ \mu^-$

Should neutral weak current only occur for $u \to u$, $d \to d$, $s \to s$ and not for the flavor-changing case $d \to s$, $s \to d$, $u \to c$, etc.?

In analogy with the charged weak current, one should have, for the weak neutral current, a term like

$$\bar{d}_c d_c \quad \text{where} \quad d_c = \cos\theta_c d + \sin\theta_c s \tag{7.36}$$

now,

$$\begin{aligned} \bar{d}_c d_c &= \left(\cos\theta_c \bar{d} + \sin\theta_c \bar{s}\right)\left(\cos\theta_c d + \sin\theta_c s\right) \\ &= \cos^2\theta_c \bar{d}d + \sin^2\theta_c \bar{s}s + \left(\sin\theta_c \cos\theta_c\right)\left(\bar{s}d + \bar{d}s\right) \end{aligned} \tag{7.37}$$

Equation 4.37 shows that the flavor-changing neutral current terms $\bar{s}d + \bar{d}s$ should exist, which is in disagreement with the tiny B.R. for $K_L \to \mu^+\mu^-$ decay.

The solution to this puzzle was the suggestion that a new quark flavor, called charm, exists. We now have two quark doublets

$$\begin{pmatrix} u \\ d \end{pmatrix} \text{ and } \begin{pmatrix} c \\ s \end{pmatrix}$$

It is natural, in an extension of the Cabbibo mixing, to expect that the weak eigenstate for S (i.e. S_c) can be written as

$$S_c = \cos\theta_c\, S - \sin\theta_c\, d \tag{7.38}$$

Therefore, the combined neutral current from d_c and S_c is

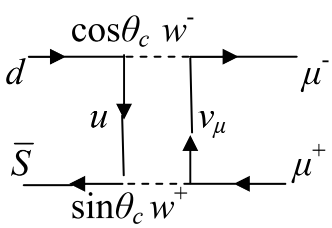
$$\bar{d}_c d_c + \bar{S}_c S_c = \bar{d}d + \bar{S}S \tag{7.39}$$

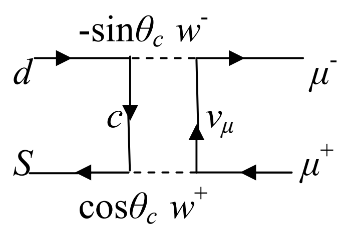
and there is no flavor-changing $\bar{d}S$, $\bar{S}d$ terms.

The presence of the charm quark also implies that the hadronic weak charged current becomes

$$J_\mu = \bar{D}_c \gamma_\mu (1-\gamma_5) U + \bar{S}_c \gamma_\mu (1-\gamma_5) C \tag{7.40}$$

One expects that second-order charge-current processes can contribute to $K_L^0 \to \mu^+ \mu^-$:



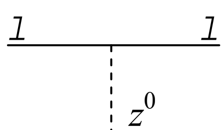


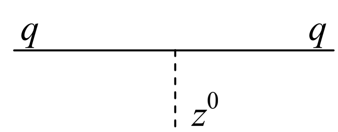
Note that these two diagrams involve an up and a charm quark exchange, respectively. If $m_u = m_c$, these two diagrams would cancel and would not contribute the $K_L^0 \to \mu^+ \mu^-$ decay.

Based on the experimentally observed B.R., one can determine that the mass of the charm quark is $m_c \simeq 1-3\ GeV$.

In 1973 Kobayashi and Maskawa extended the Cabbibo theory to 3 generations. In this case, there are four parameters in the 3 x 3 matrix; three of them are the mixing angles θ_1, θ_2, θ_3 plus one phase δ ($e^{i\delta}$). A non-zero value of S implies CP-violation in charged-current weak interactions.

The electroweak theory of Glashow, Weinberg and Salam in the 1960s, which unified the electromagnetic and the weak interactions, had a unique prediction on the existence of neutral current:





However, it is difficult to identify effects of neutral current in the quark sector, since the NC effect is overshadowed by the electromagnetic process. To overcome this problem, one can use neutrino beam, which is not subjected to EM interaction.

In 1973, neutral current was discovered in CERN using a Freon bubble chamber (Gargamelle). The NC reaction

$$\nu_\mu\left(\overline{\nu}_\mu\right)+N\to\nu_\mu\left(\overline{\nu}_\mu\right)+\text{hadrons}$$
$$\text{(no } \mu^- \text{ or } \mu^+\text{)}$$

was observed. The CC events from

$$\nu_\mu\left(\overline{\nu}_\mu\right)+N\to\mu^-\left(\mu^+\right)+\text{hadron}$$

were also observed. The ratio of the neutral current yield versus charged current yield was found to be

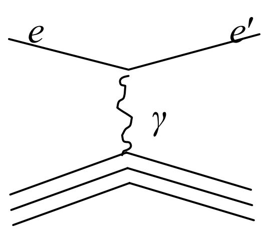
$$\left(NC/CC\right)_\nu=0.21\pm0.03$$
$$\left(NC/CC\right)_{\overline{\nu}}=0.45\pm0.09$$

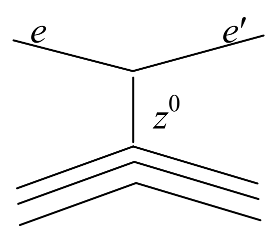
The standard model prediction is

$$\begin{aligned}R_\nu&=\left(\frac{NC}{CC}\right)_\nu=\frac{1}{2}-\sin^2\theta_w+\frac{20}{27}\sin^4\theta_w\\R_{\overline{\nu}}&=\left(\frac{NC}{CC}\right)_{\overline{\nu}}=\frac{1}{2}-\sin^2\theta_w+\frac{20}{9}\sin^4\theta_w\end{aligned}\tag{7.41}$$

Hence one deduces $0.3<\sin^2\theta_w<0.4$ from the Gargamelle data.

In 1978, evidence for neutral current was found also at SLAC in a deep-inelastic scattering experiment using polarized electron beam. In this $\vec{e}+d\to\vec{e}\,'+x$ scattering, the following two diagrams can contribute:





The single-spin asymmetry, A, defined as

$$A=\frac{\sigma_R-\sigma_L}{\sigma_R+\sigma_L}\tag{7.42}$$

could have a non-zero value as a result of the interference of the γ and z^0 diagrams. This asymmetry is due to a term proportional to $\vec{\sigma}_e \bullet \vec{P}_e$, where $\vec{\sigma}_e$ is the spin of the electron (R, L refers to the right-handed and left-handed electron beam, respectively). $\vec{P}_e$ is the momentum vector of the scattered electron. It is clear that the $\vec{\sigma}_e \bullet \vec{P}_e$ term violates parity.

The predicted asymmetry is related to the Weinberg angle θ_w:

$$A = -\frac{G_F Q^2}{2\sqrt{2}\pi\alpha}\frac{9}{10}\left\{1 - \frac{20}{9}\sin^2\theta_w + \left(1 - 4\sin^2\theta_w\right)\left[\frac{1-(1-y)^2}{1+(1-y)^2}\right]\right\} \tag{7.43}$$

The experiment obtained

$$\frac{A}{Q^2} = \left(-9.5 \pm 1.6\right) \times 10^{-5}$$

which implies

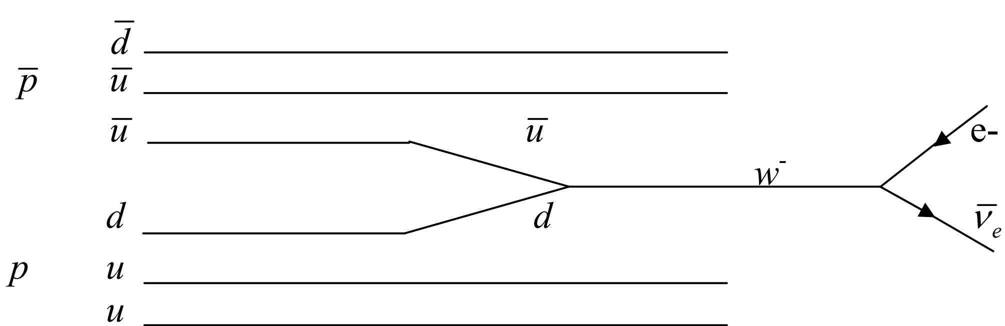
$$\sin^2\theta_w = 0.20 \pm 0.03$$

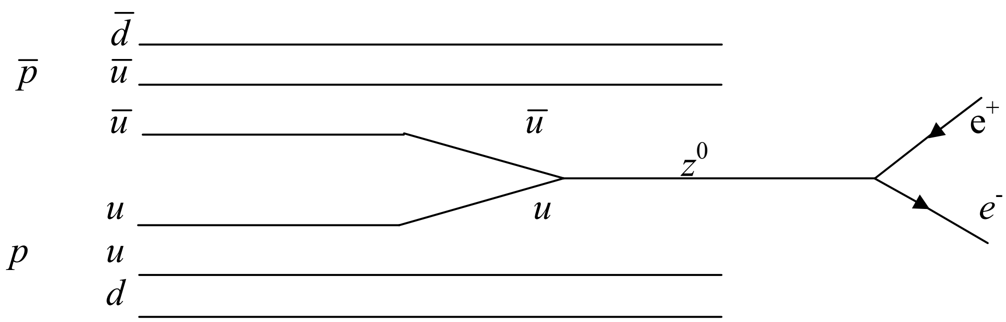
In the electro-weak theory, the masses of w and z can be determined once $\sin^2\theta_w$ is known:

$$\begin{aligned} M_w^2 &= \frac{\pi\alpha}{\sqrt{2}G_F \sin^2\theta_w} \\ M_z^2 &= M_w^2 / \cos^2\theta_w \end{aligned} \tag{7.44}$$

with $\sin^2\theta_w = 0.23$, Equation 7.44 predicts $M_w = 80$ GeV and $M_z = 92$ GeV.

The w and z bosons were observed in $p\bar{p}$ collision at CERN in 1982:





The observation of w/z^0 and their production and decay characteristics dramatically confirmed the electroweak theory.

Another important ingredient in the standard model, namely the Higgs particle, remains to be found.

There are numerous examples of weak interactions and weak decays. They can be characterized as proceeding via neutral current (NC), charged current (CC), or charged current plus neutral current (NC + CC). Also, they can be classified according to whether they are purely leptonic, semi-leptonic, or non-leptonic. An incomplete list follows:

	Charged Current	Neutral Current	Charged/Neutral
Leptonic	$\mu \to e\nu\bar{\nu}$ $\tau \to \ell\nu\bar{\nu}$ $\nu_\mu e \to \mu\nu e$	$\nu_\mu e \to \nu_\mu e$ $e^+e^- \to \ell^+\ell^-$	$\nu_e e \to \nu_e e$ $e^+e^- \to \nu_e\bar{\nu}_e$
Semi-leptonic	$\pi \to \mu\nu$ $D \to K\ell\nu$ $n \to pe^-\bar{\nu}$ $\nu_\mu N \to \mu^- x$	$\nu N \to \nu N$ $\bar{\nu}_e + D \to \bar{\nu}_e + n + p$	
Non-leptonic	$K \to \pi\pi$ $D \to K\pi$ $\Lambda \to p\pi^-$	$pp \to pp$	$pn \to pn$

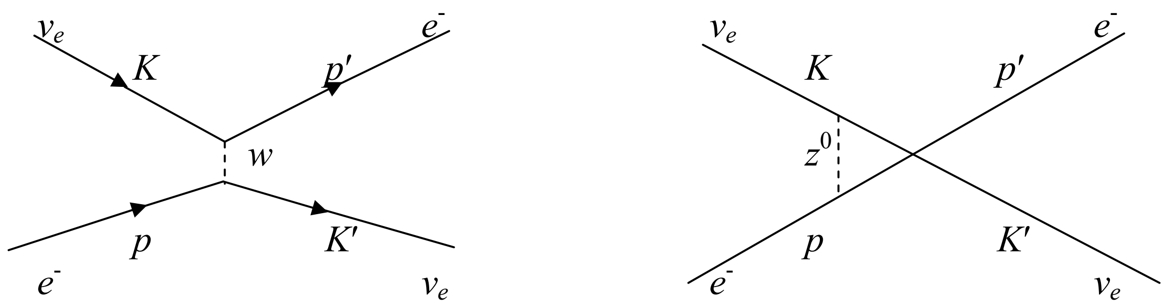
In purely leptonic processes, only leptons appear in the interactions or decays. For semi-leptonic processes, both hadrons and leptons participate. In non-leptonic processes, only hadrons appear.

Pure Leptonic Weak Interaction

We consider the following reaction:

$$\nu_e e^- \rightarrow e^- \nu_e$$

This reaction can proceed via charged current as well as neutral current:



This reaction was used to detect solar neutrino in several water Cherenkov detector experiments.

We now consider the charged-current contribution to this process. At low energy and intermediate energy, it is appropriate to adopt Fermi's contact current-current interaction. The invariant amplitude is

$$M = \frac{G}{\sqrt{2}}\left(\bar{u}(K')\gamma^{\mu}\left(1-\gamma^5\right)u(p)\right)\left(\bar{u}(p')\gamma_{\mu}\left(1-\gamma^5\right)u(K)\right) \qquad (7.45)$$

To evaluate $|M|^2$, one needs M^* which contains adjoint V-A current such as

$$\begin{aligned}&\left[\bar{u}(K')\gamma^{\mu}\left(1-\gamma^5\right)u(p)\right]^* = \left[\bar{u}(K')\gamma^{\mu}\left(1-\gamma^5\right)u(p)\right]^{\dagger}\\ &= \bar{u}(p)\gamma^{\mu}\left(1-\gamma^5\right)u(K')\end{aligned} \qquad (7.46)$$

(Note that for S-P current, we have $\left[\bar{u}(K')\left(1-\gamma^5\right)u(p)\right]^* = \bar{u}(p)\left(1+\gamma^5\right)u(K')$)

$$\overline{|M|^2} = \frac{1}{2}\sum_{spin}|M|^2 = \frac{G^2}{4} Tr\left(\gamma^\mu\left(1-\gamma^5\right)\not{p}\gamma^\nu\left(1-\gamma^5\right)\not{K}'\right) \times Tr\left(\gamma_\mu\left(1-\gamma^5\right)\not{K}\gamma_\nu\left(1-\gamma^5\right)\not{p}'\right) \tag{7.47}$$

Note that in Equation 7.47, a factor of ½ is used instead of ¼ for the average of the initial spin states since ν_e is left-handed and is in a unique spin state (the electron is unpolarized and can be in either spin states).

Noting $\quad Tr\left(\gamma^\mu\left(1-\gamma^5\right)\not{p}\gamma^\nu\left(1-\gamma^5\right)\not{K}'\right) = 2Tr\left(\gamma^\mu \not{p}\gamma^\nu\left(1-\gamma^5\right)\not{K}'\right)$

Equation 7.47 becomes

$$\overline{|M|^2} = G^2 Tr\left(\gamma^\mu \not{p}\gamma^\nu\left(1-\gamma^5\right)\not{K}'\right) Tr\left(\gamma_\mu \not{K}\gamma_\nu\left(1-\gamma^5\right)\not{p}'\right) \tag{7.48}$$

Some useful trace theorems are listed below:

$$Tr\left(\gamma^\mu \not{P}_1\gamma^\nu \not{P}_2\right) = 4\left(P_1^\mu P_2^\nu + P_1^\nu P_2^\mu - \left(P_1 \bullet P_2\right)g^{\mu\nu}\right) \tag{7.49}$$

$$Tr\left(\gamma^\mu \not{P}_1\gamma^\nu\gamma^5 \not{P}_2\right) = 4i\varepsilon^{\mu\alpha\nu\beta}P_{1\alpha}P_{2\beta} \tag{7.50}$$

($\varepsilon^{\mu\alpha\nu\beta}$ is antisymmetric tensor for $0 \le \varepsilon,\ \alpha,\ \gamma,\ \beta \le 3$, $\varepsilon^{0123} = -1$ and it changes sign upon permutation)

Equations 7.49 and 7.50 give

$$Tr\left(\gamma^\mu \not{P}_1\gamma^\nu \not{P}_2\right)Tr\left(\gamma_\mu \not{P}_3\gamma_\nu \not{P}_4\right) = 32\left[\left(P_1 \bullet P_3\right)\left(P_2 \bullet P_4\right) + \left(P_1 \bullet P_4\right)\left(P_2 \bullet P_3\right)\right] \tag{7.51}$$

and

$$Tr\left(\gamma^\mu \not{P}_1\gamma^\nu\gamma^5 \not{P}_2\right)Tr\left(\gamma_\mu \not{P}_3\gamma_\nu \not{P}_4\right) = 32\left[\left(P_1 \bullet P_3\right)\left(P_2 \bullet P_4\right) - \left(P_1 \bullet P_4\right)\left(P_2 \bullet P_3\right)\right] \tag{7.52}$$

(note that $\varepsilon^{\mu\nu\lambda\sigma}\varepsilon_{\mu\nu\kappa\tau} = -2\left(\delta_\kappa^\lambda\delta_\tau^\sigma - \delta_\tau^\lambda\delta_\kappa^\sigma\right)$)

Note that

$$Tr\left(\gamma^{\mu} \underbrace{\not{P}_1}_{\substack{\text{symmetric WRT } \mu\nu \\ \text{interchange}}} \gamma^{\nu} \not{P}_2\right) Tr\left(\gamma_{\mu} \not{P}_3 \underbrace{\gamma_{\nu}}_{\substack{\text{antisymmetric WRT } \mu\nu \\ \text{interchange}}} \gamma^5 \not{P}_4\right) = 0 \tag{7.53}$$

using Equations 7.51, 7.52, and 7.53 Equation 7.48 becomes

$$\begin{aligned} \overline{|M|^2} &= 64G^2 (K \bullet P)(K' \bullet P') \\ &= 16G^2 S^2 \qquad (\text{ignoring electron's mass}) \end{aligned} \tag{7.54}$$

In the C.M. frame, the differential cross-section is

$$\frac{d\sigma}{d\Omega}\left(\nu_e e^- \rightarrow e^- \nu_e\right) = \frac{1}{64\pi^2 S} \overline{|M|^2} = \frac{G^2 S}{4\pi^2} \tag{7.55}$$

The angular distribution is isotropic, and the total cross-section is

$$\sigma\left(\nu_e e^- \rightarrow e^- \nu_e\right) = \frac{G^2 S}{\pi} \tag{7.56}$$

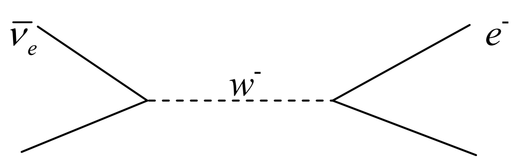
Note that in the lab frame, the angular distribution is no longer isotropic, due to the boost. Hence the $\nu_e e^- \rightarrow e^- \nu_e$ reaction can still be used to isolate ν_e originating from the sun (since e^- tends to move along the same direction as the incoming ν_e) .

The cross-sections for the $\nu_\mu e^- \rightarrow \mu^- \nu_e$ reaction, which can only proceed via charged current interaction, are identical to the CC part of $\nu_e e^- \rightarrow e^- \nu_e$ and are given by Equations 7.55 and 7.56.

We now consider another related reaction

$$\bar{\nu}_e e^- \rightarrow e^- \bar{\nu}_e$$

The reaction proceeds via an intermediate w boson



e^- $\bar{\nu}_e$

The $\bar{\nu}_e e^- \to e^- \bar{\nu}_e$ is related to $\nu_e e^- \to e^- \nu_e$ via crossing symmetry.

Interchanging ν_e's in the initial and final states in $\nu_e e^- \to e^- \nu_e$ would lead to $\bar{\nu}_e e^- \to e^- \bar{\nu}_e$. ($P_A \leftrightarrow -P_D$)

$$S' = \left(P'_A + P'_B\right)^2 = \left(-P_D + P_B\right)^2 = \left(P_C - P_A\right)^2 = t$$
$$t' = \left(P'_A - P'_C\right)^2 = \left(P_D + P_C\right)^2 = \left(P_A + P_B\right)^2 = S$$

Hence, $s \leftrightarrow t$ would relate $\nu_e e^- \to e^- \nu_e$ to $\bar{\nu}_e e^- \to e^- \bar{\nu}_e$. Interchanging $s \leftrightarrow t$, Equation 7.54 becomes

$$\overline{|M|^2} = 16G^2 t^2 = 4G^2 S^2 \left(1 - \cos\theta\right)^2 \tag{7.57}$$
$$\left(\text{since } t = -\frac{S}{2}\left(1 - \cos\theta\right)\right)$$

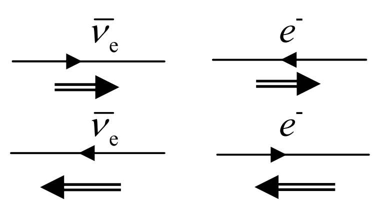
The differential cross-section in the C.M. frame is

$$\frac{d\sigma}{d\Omega}\left(\bar{\nu}_e e^- \to e^- \bar{\nu}_e\right) = \frac{G^2 S}{16\pi^2}\left(1 - \cos\theta\right)^2 \tag{7.58}$$

and the total cross-section is

$$\sigma\left(\bar{\nu}_e e^- \to e^- \bar{\nu}_e\right) = \frac{G^2 S}{3\pi} = \frac{1}{3}\sigma\left(\nu_e e^- \to e^- \nu_e\right) \tag{7.59}$$

Equation 7.58 shows that the reaction is backward-peaked and the cross-section vanishes at $\theta = 0^{\circ}$. This can be readily understood from helicity consideration. In the C.M. frame



in the initial state (7.60)

in the final state for $\theta_{\text{C.M.}} = 0$

Angular momentum conservation prohibits scattering to $\theta_{\text{C.M.}} = 0$

An intuitive interpretation for the factor ⅓ appearing in Equation 7.59 is that the figure in 7.60 shows that only the +1 of the three helicity states (+1,0,-1) of w can participate in the $\bar{\nu}_e e^- \to e^- \bar{\nu}_e$ reaction.

Another reaction closely related to the $\nu_e e^- \to e^- \nu_e$ is

$$e^+ e^- \to \bar{\nu}_e \nu_e$$

This reaction can be obtained from $\nu_e e^- \to e^- \nu_e$ by crossing ν_e with e^-, i.e. $P_A \leftrightarrow -P_C$

Therefore, $$S' = \left(P'_A + P'_B\right)^2 = \left(-P_C + P_B\right)^2 = \left(P_A - P_D\right)^2 = u$$

Interchanging $s \leftrightarrow u$ in Equation 7.54, we have

$$\overline{|M|^2} = 16G^2u^2 = 4G^2S^2\left(1+\cos\theta\right)^2 \tag{7.61}$$

$$\left(\text{since } u = \frac{-s}{2}\left(1+\cos\theta\right)\right)$$

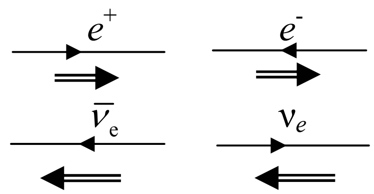
The differential cross-section is

$$\frac{d\sigma}{d\Omega}\left(e^+e^- \to \bar{\nu}_e\nu_e\right) = \frac{G^2 s}{16\pi^2}\left(1+\cos\theta\right)^2 \tag{7.62}$$

and the total cross-section is

$$\sigma\left(e^+e^- \to \bar{\nu}_e\nu_e\right) = \frac{G^2 s}{3\pi} = \sigma\left(\bar{\nu}_e e^- \to e^- \bar{\nu}_e\right) = \frac{1}{3}\sigma\left(\nu_e e^- \to e^- \nu_e\right) \tag{7.63}$$

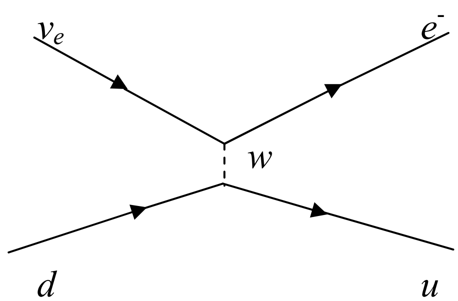
Equation 7.62 shows that $\bar{\nu}_e$ in the $e^+e^- \rightarrow \bar{\nu}_e\nu_e$ can not go to 180^o. Again, this can be understood by considering angular momentum conservation:

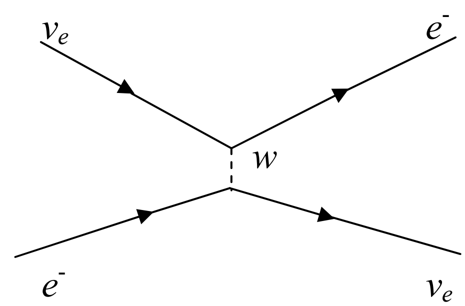


in the initial state

in the final state for $\theta_{C.M.} = 180^o$, and is not allowed due to angular momentum conservation

Another reaction closely related to $\nu_e e^- \rightarrow e^- \nu_e$ is the $\nu_e d \rightarrow e^- u$ reaction





The $\nu_e d \rightarrow e^- u$ is a semi-leptonic process, but the cross-section is almost identical to that of $\nu_e e^- \rightarrow e^- \nu_e$. The only difference is that $\nu_e d \rightarrow e^- u$ also contains the $\cos^2\theta_c$ term to account for the mixing between d and s.

Similarly, one can show that the $\nu_e u \rightarrow e^+ d$ reaction is the analog of $\bar{\nu}_e\nu_e \rightarrow e^+e^-$ reaction.

We can summarize the above discussion with the following table. If one considers only the charged current and set the Cabbibo angle θ_c to 0, then

$$
\begin{array}{lll}
\dfrac{d\sigma}{d\Omega}=\dfrac{G^2 s}{4\pi^2} & \dfrac{d\sigma}{d\Omega}=\dfrac{G^2 s}{16\pi^2}\left(1-\cos\theta\right)^2 & \dfrac{d\sigma}{d\Omega}=\dfrac{G^2 s}{16\pi^2}\left(1+\cos\theta\right)^2 \\
\hline
\nu_e e^- \to e^- \nu_e & \bar{\nu}_e e^- \to e^- \bar{\nu}_e & e^+ e^- \to \bar{\nu}_e \nu_e \\
\nu_\mu e^- \to \mu^- \nu_e & \bar{\nu}_e e^- \to \mu^- \bar{\nu}_\mu & \mu^+ e^- \to \bar{\nu}_\mu \nu_e \\
\nu_e d \to e^- u & & \bar{\nu}_e u \to e^+ d \\
\bar{\nu}_e \bar{d} \to e^+ \bar{u} & & \nu_e \bar{u} \to e^- \bar{d} \\
\nu_\mu d \to \mu^- u & & \bar{\nu}_\mu u \to \mu^+ d \\
\bar{\nu}_\mu \bar{d} \to \mu^+ \bar{u} & & \nu_\mu \bar{u} \to \mu^- \bar{d}
\end{array}
\tag{7.64}
$$

Note that in 7.64, an isotropic angular distribution is obtained if the initial colliding pair have identical helicities (both are left-handed, or both are right-handed). If they have opposite helicity, then the cross-section is anisotropic and the integrated cross-section drops to ⅓ of the isotropic reactions.

The table in 7.64 shows that there are no interactions between $\nu_\mu u,\ \nu_\mu \bar{d},\ \bar{\nu}_\mu d,\ \bar{\nu}_\mu \bar{u}$.

Neutrino-Induced Deep-Inelastic Scattering (DIS)

The underlying processes for neutrino induced DIS off a nucleon include:

$$
\begin{aligned}
&\frac{d\sigma}{d\Omega}\left(\nu_\mu d \to \mu^- u\right)=\frac{G^2 s}{4\pi^2} \\
&\frac{d\sigma}{d\Omega}\left(\bar{\nu}_\mu u \to \mu^+ d\right)=\frac{G^2 s}{16\pi^2}\left(1+\cos\theta\right)^2
\end{aligned}
\tag{7.65}
$$

It is useful to express Equation 7.65 in terms of Lorentz invariant quantities such as y, where $y=\dfrac{p\bullet q}{p\bullet K}$. In the c.m. frame, neglecting masses, we have

$$y = \frac{p \bullet q}{p \bullet K} = \frac{p \bullet (K - K')}{p \bullet K} = 1 - \frac{p \bullet K'}{p \bullet K} = 1 - \frac{1}{2}(1 + \cos\theta) \tag{7.66}$$

Hence, $1 + \cos\theta = 2(1 - y)$

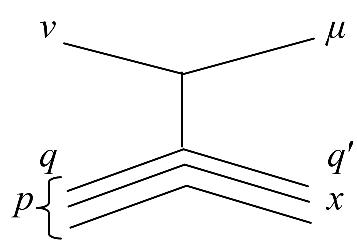
Also,
$$\begin{aligned} d\Omega &= 2\pi \sin\theta d\theta = -2\pi d\cos\theta = 4\pi dy \\ \frac{d\sigma}{d\Omega} &= \frac{1}{4\pi}\frac{d\sigma}{dy} \end{aligned} \tag{7.67}$$

Equation 7.65 should be written as

$$\begin{aligned} \frac{d\sigma}{d\Omega}\left(\nu_\mu d \to \mu^- u\right) &= \frac{G^2 \hat{s}}{4\pi^2} \\ \frac{d\sigma}{d\Omega}\left(\bar{\nu}_\mu u \to \mu^+ d\right) &= \frac{G^2 s}{16\pi^2}(1 + \cos\theta)^2 \end{aligned} \tag{7.68}$$

where $\hat{s}$ represents the Mandelstam parameter s in the $\nu_\mu d$ system. Similarly, one can define $\hat{t}$ and $\hat{u}$.

For a DIS process



$\hat{s}$, $\hat{t}$, $\hat{u}$ refer to the $\nu + q \to \mu + q'$ subprocess, and s, t, u refer to the $\nu + N \to \mu + x$ process.

$$\hat{s} = \left(\hat{P}_a + \hat{P}_b\right)^2 \simeq 2\hat{P}_a \bullet \hat{P}_b = 2\left(P_a\right) \bullet \left(xP_b\right) = x \bullet 2P_a \bullet P_b = xs \tag{7.69}$$

where x is the fraction of proton's momentum carried by the quark. Similarly, one can show that $\hat{t} = t;\ \hat{u} = xu$

Using Equations 7.66, 7.67, and 7.69, Equation 7.68 becomes

$$\frac{d\sigma}{dy}\left(\nu_\mu d \to \mu^- u\right) = \frac{G^2 xs}{\pi} \qquad \text{(same for } \bar{\nu}_\mu \bar{d} \to \mu^+ \bar{u}\text{)}$$
$$\frac{d\sigma}{dy}\left(\bar{\nu}_\mu u \to \mu^+ d\right) = \frac{G^2 xs}{\pi}(1-y)^2 \qquad \text{(same for } \nu_\mu \bar{u} \to \mu^- \bar{d}\text{)} \tag{7.70}$$

Equation 7.70 corresponds to DIS on the quark which carries a fraction x of the nucleon's momentum. For DIS on a nucleon, one needs to take into account the probability that the quark carries a momentum fraction x. Hence

$$\frac{d\sigma}{dxdy}\left(\nu_\mu p \to \mu^- x\right) = \frac{G^2 xs}{\pi}\left[d_p(x) + (1-y)^2 \bar{u}_p(x)\right] \tag{7.71}$$

where the scattering off an antiquark is also considered.

(Note that in Equation 7.71 there is no e_q^2 factor, since it is a weak interaction and the coupling is not proportional to e.)

Similarly, for a DIS off a neutron, we have

$$\frac{d\sigma}{dxdy}\left(\nu_\mu n \to \mu^+ x\right) = \frac{G^2 xs}{\pi}\left[d_n(x) + (1-y)^2 \bar{u}_n(x)\right] \tag{7.72}$$

Isospin symmetry demands $d_n(x) = u_p(x)$; $\bar{u}_n(x) = \bar{d}_p(x)$

For a scattering off an isoscalar target, which has an equal number of protons and neutrons (like d, ^{12}C, ^{40}Ca, . . .), the DIS cross-section per nucleon is an average of Equations 7.71 and 7.72:

$$\frac{d\sigma}{dxdy}\left(\nu_\mu N \to \mu^- x\right) = \frac{G^2 xs}{2\pi}\left[\left(u_p(x) + d_p(x)\right) + (1-y)^2\left(\bar{u}_p(x) + \bar{d}_p(x)\right)\right] \tag{7.73}$$

Similarly,

$$\frac{d\sigma}{dxdy}\left(\bar{\nu}_\mu N \to \mu^+ x\right) = \frac{G^2 xs}{2\pi}\left[\left(\bar{u}_p(x) + \bar{d}_p(x)\right) + (1-y)^2\left(u_p(x) + d_p(x)\right)\right] \tag{7.74}$$

Note that the $\bar{\nu}_\mu N$ DIS is obtained from the $\nu_\mu N$ DIS by $s \leftrightarrow u$ interchange (or $1 \leftrightarrow (1-y)^2$ interchange).

Equations 7.73 and 7.74 show that a comparison between

$$\frac{d\sigma}{dxdy}\left(\nu_\mu N \to \mu^- x\right)$$

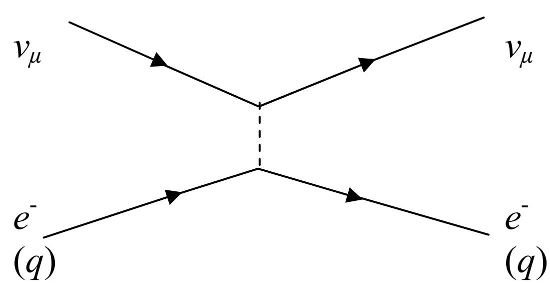
and

$$\frac{d\sigma}{dxdy}\left(\bar{\nu}_\mu N \to \mu^+ x\right)$$

allows a separation of $Q(x) = u(x) + d(x)$ from the antiquark distribution $\bar{Q}(x) = \bar{u}(x) + \bar{d}(x)$.

Neutral-Current Weak Interaction

For processes such as $\nu_\mu e^- \to \nu_\mu e^-$ and $\bar{\nu}_\mu e^+ \to \bar{\nu}_\mu e^+$, only neutral-current contributes. Similarly for reactions $\nu_\mu q \to \nu_\mu q$ and $\bar{\nu}_\mu q \to \bar{\nu}_\mu q$.



The invariant matrix element for $\nu_\mu q \to \nu_\mu q$ can be written as

$$M = \frac{G}{\sqrt{2}}\left[\bar{u}_\nu \gamma^\mu\left(1-\gamma^5\right)u_\nu\right]\left[\bar{u}_q \gamma_\mu\left(C_V^q - C_A^q\gamma^5\right)u_q\right] \tag{7.75}$$

M contains two terms:

$$\nu_L q_L \to \nu_L q_L \text{ and } \nu_L q_R \to \nu_L q_R$$

Note that Equation 7.75 shows that the neutral-current for neutrino is purely left-handed (since only left-handed neutrino is known to exist). For quarks, the neutral current can be a mixture of $V - A$ and $V + A$ currents:

$$\bar{u}_q \gamma_\mu \left(C_V^q - C_A^q \gamma^5 \right) u_q = \bar{u}_q \gamma_\mu \underbrace{g_L^q}_{\text{left-handed}} \left(1 - \gamma^5\right) u_q + \bar{u}_q \gamma_\mu \underbrace{g_R^q}_{\text{right-handed}} \left(1 + \gamma^5\right) u_q \tag{7.76}$$

where

$$g_L^q = \frac{1}{2}\left(C_V^q + C_A^q \right) \text{ and } g_R^q = \frac{1}{2}\left(C_V^q - C_A^q \right) \tag{7.77}$$

represent the V- A and the $V + A$ component of the neutral current, respectively.

Note that for $V - A$ coupling: q is left-handed, $\bar{q}$ is right-handed
for $V + A$ coupling: q is right-handed, $\bar{q}$ is left-handed
(see Equations 5.103 and 5.107)

The expression for ν-induced neutral-current DIS on an isoscalar target is

$$\begin{aligned} \frac{d\sigma}{dxdy}(\nu N \to \nu x) = \frac{G^2 xs}{2\pi} \Big\{ & g_L^2 \left(Q(x) + (1-y)^2 \bar{Q}(x) \right) \\ & + g_R^2 \left(\bar{Q}(x) + (1-y)^2 Q(x) \right) \Big\} \end{aligned} \tag{7.78}$$

For $\bar{\nu} N \to \bar{\nu} x$, one interchanges $s \leftrightarrow u$ (or $1 \leftrightarrow (1 - y)^2$).

$$\begin{aligned} \frac{d\sigma^{NC}}{dxdy}(\bar{\nu} N \to \bar{\nu} x) = \frac{G^2 xs}{2\pi} \Big\{ & g_L^2 \left(\bar{Q}(x) + (1-y)^2 Q(x) \right) \\ & + g_R^2 \left(Q(x) + (1-y)^2 \bar{Q}(x) \right) \Big\} \end{aligned} \tag{7.79}$$

where

$$g_L^2 = \left(g_L^u\right)^2 + \left(g_L^d\right)^2 \qquad g_R^2 = \left(g_R^u\right)^2 + \left(g_R^d\right)^2 \tag{7.80}$$

In the electro-weak theory, the values of the vector coupling C_V^f and axial-coupling C_A^f are given as

$$C_V^f = T_f^3 - 2\sin^2\theta_w Q_f \tag{7.81}$$

$$C_A^f = T_f^3 \tag{7.82}$$

where Q_f is the electric charge of the fermion, and T_f^3 is the third component of the weak isospin of the fermion. Leptons and quarks form weak-isospin doublets as follows:

$$\begin{pmatrix} \nu_e \\ e^- \end{pmatrix}\begin{pmatrix} \nu_\mu \\ \mu^- \end{pmatrix}\begin{pmatrix} \nu_\tau \\ \tau^- \end{pmatrix}\begin{pmatrix} u \\ d \end{pmatrix}\begin{pmatrix} c \\ s \end{pmatrix}\begin{pmatrix} t \\ b \end{pmatrix}$$

$T_f^3 = \frac{1}{2}$ for the upper members of the doublets and $T_f^3 = -\frac{1}{2}$ for the lower members.

Table 7.83

	Q_f	T_f^3	C_A^f	C_V^f
ν_e, ν_μ, ν_τ	0	$+½$	½	½
e^-, μ^-, τ^-	-1	$-1/2$	$-1/2$	$-1/2 + 2\sin^2\theta_w$ $(\sim\ -0.03)$
u, c, t	$+2/3$	$+1/2$	$+1/2$	$1/2 - 4/3\sin^2\theta_w$ $(\sim\ 0.19)$
d, s, b	$-1/3$	$-1/2$	$-1/2$	$-1/2 + 2/3\sin^2\theta_w$ $(\sim\ -0.34)$

In table 7.83, the values for C_V^f were calculated using $\sin^2\theta_w = 0.231$, determined from experiments.

This table shows that for neutrinos, $C_A = C_V = ½$, and $g_L^\nu = 1/2$, $g_R^\nu = 0$, reflecting that ν is left-handed.

It also shows that for e^-, μ^-, τ^-, $C_V \simeq 0$, $C_A = -1/2$, hence the neutral current in this case is almost purely an axial-vector coupling, with $g_L^e \simeq -1/4$, $g_R^e \simeq +1/4$.

A comparison between Equations 7.74 and 7.79 shows that the neutral current cross-section reduces to the charged current cross-section when

$$g_L = 1, g_R = 0$$

We now revisit $\nu - e$ scattering taking into account the contribution of neutral current.

First, we consider the $\nu_\mu e^- \rightarrow \nu_\mu e^-$ and $\bar{\nu}_\mu e^- \rightarrow \bar{\nu}_\mu e^-$ reactions, which can only proceed via neutral current

$$m^{NC}\left(\nu_\mu e \rightarrow \nu_\mu e\right) = \frac{G_N}{\sqrt{2}}\left(\bar{u}_\nu \gamma^\mu \left(1-\gamma^5\right) u_\nu\right) \left(\bar{u}_e \gamma_\mu \left(C_V^e - C_A^e \gamma^5\right) u_e\right) \tag{7.84}$$

where $$G_N = G_F = G$$

Following procedures analogous to Equations 7.45 and 7.54, one obtains

$$\begin{aligned} \frac{d\sigma}{dy}\left(\nu_\mu e \rightarrow \nu_\mu e\right) &= \frac{G^2 s}{\pi}\left[g_L^2 + g_R^2 (1-y)^2\right] \\ &= \frac{G^2 s}{4\pi}\left[\left(C_V^e + C_A^e\right)^2 + \left(C_V^e - C_A^e\right)^2 (1-y)^2\right] \end{aligned} \tag{7.85}$$

$$\frac{d\sigma}{dy}\left(\bar{\nu}_\mu e \rightarrow \bar{\nu}_\mu e\right) = \frac{G^2 s}{4\pi}\left[\left(C_V^e + C_A^e\right)^2 (1-y)^2 + \left(C_V^e - C_A^e\right)^2\right] \tag{7.86}$$

Note that Equation 7.86 is obtained from Equation 7.85 by $s \leftrightarrow u$ interchange (or $1 \leftrightarrow (1-y)^2$). These two cross-sections are also related by $C_A^e \leftrightarrow -C_A^e$ interchange.

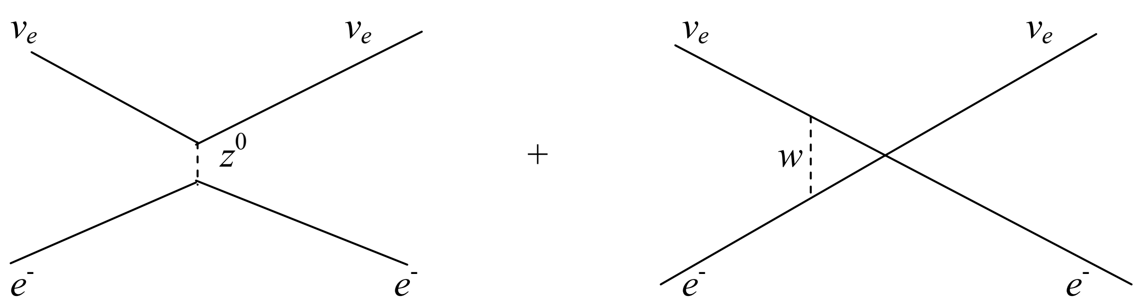
Integrating over y, Equations 7.85 and 7.86 become

$$\sigma\left(\nu_\mu e \to \nu_\mu e\right) = \frac{G^2 s}{3\pi}\left(C_V^{e^2} + C_V^e C_A^e + C_A^{e^2}\right) \tag{7.87}$$

$$\sigma\left(\bar{\nu}_\mu e \to \bar{\nu}_\mu e\right) = \frac{G^2 s}{3\pi}\left(C_V^{e^2} - C_V^e C_A^e + C_A^{e^2}\right) \tag{7.88}$$

Since $C_v^e \sim 0$ (Table 7.83), we expect that $\sigma\left(\nu_\mu e \to \nu_\mu e\right) \simeq \sigma\left(\bar{\nu}_\mu e \to \bar{\nu}_\mu e\right)$ (recall that $\sigma^{cc}\left(\nu_e e^- \to \nu_e e^-\right) = 3\sigma^{cc}\left(\bar{\nu}_e e^- \to \bar{\nu}_e e^-\right)$).

The $\nu_e e^- \to \nu_e e^-$ reaction contains contributions from neutral current as well as charged current:



The corresponding amplitudes are

$$M^{NC}\left(\nu_e e \to \nu_e e\right) = \frac{G}{\sqrt{2}}\left(\bar{\nu}\gamma^\mu\left(1-\gamma^5\right)\nu\right)\left(\bar{e}\gamma_\mu\left(C_v^e - C_A^e\gamma^5\right)e\right) \tag{7.89}$$

$$\begin{aligned} M^{CC}\left(\nu_e e \to \nu_e e\right) &= -\frac{G}{\sqrt{2}}\left(\bar{e}\gamma^\mu\left(1-\gamma^5\right)\nu\right)\left(\bar{\nu}\gamma_\mu\left(1-\gamma^5\right)e\right) \\ &= \frac{G}{\sqrt{2}}\left(\bar{\nu}\gamma^\mu\left(1-\gamma^5\right)\nu\right)\left(\bar{e}\gamma_\mu\left(1-\gamma^5\right)e\right) \end{aligned} \tag{7.90}$$

The negative sign for the CC diagram is due to the interchange of the outgoing fermions.

The second line in Equation 7.90 is obtained using the Fierz transformation, which relates the 'charge-exchange ordering' to 'charge-retention ordering'. (For a derivation of the Fierz theorems, see "Electroweak Interactions" by Peter Renton, Appendix E.)

Adding the NC and the CC contributions, one has

$$M^{NC} + M^{CC} = \frac{G}{\sqrt{2}}\left(\bar{\nu}\gamma^{\mu}\left(1-\gamma^{5}\right)\nu\right)\left(\bar{e}\gamma_{\mu}\left[\left(C_V^e+1\right)-\left(C_A^e+1\right)\gamma^{5}\right]e\right) \quad (7.91)$$

Equation 7.91 has a form analogous to Equation 7.84, except that

$$C_V^e \to C_V^{e'} = C_V^e + 1 \qquad C_A^e \to C_A^{e'} = C_A^e + 1 \quad (7.92)$$

From Equations 7.87 and 7.92, we have

$$\frac{\sigma^{NC+CC}\left(\nu_e e \to \nu_e e\right)}{\sigma^{NC}\left(\nu_\mu e \to \nu_\mu e\right)} = \frac{\left(C_V^e+1\right)^2 + \left(C_V^e+1\right)\left(C_A^e+1\right) + \left(C_A^e+1\right)^2}{\left(C_V^e\right)^2 + C_V^e C_A^e + \left(C_A^e\right)^2} \quad (7.93)$$

Similarly for $\bar{\nu}_e e \to \bar{\nu}_e e$, we have

$$\frac{\sigma\left(\bar{\nu}_e e \to \bar{\nu}_e e\right)}{\sigma\left(\bar{\nu}_\mu e \to \bar{\nu}_\mu e\right)} = \frac{\left(C_V^e+1\right)^2 - \left(C_V^e+1\right)\left(C_A^e+1\right) + \left(C_A^e+1\right)^2}{\left(C_V^e\right)^2 - C_V^e C_A^e + \left(C_A^e\right)^2} \quad (7.94)$$

Summarizing:

Reaction	Current	Cross-section is proportional to
$\nu_\mu e^- \to \nu_\mu e^-$	NC	$\left(C_V^e\right)^2 + C_V^e C_A^e + \left(C_A^e\right)^2$
$\bar{\nu}_\mu e^- \to \bar{\nu}_\mu e^-$	NC	$\left(C_V^e\right)^2 - C_V^e C_A^e + \left(C_A^e\right)^2$
$\nu_e e^- \to \nu_e e^-$	$NC + CC$	$\left(C_V^e+1\right)^2 + \left(C_V^e+1\right)\left(C_A^e+1\right) + \left(C_A^e+1\right)^2$
$\bar{\nu}_e e^- \to \bar{\nu}_e e^-$	$NC + CC$	$\left(C_V^e+1\right)^2 - \left(C_V^e+1\right)\left(C_A^e+1\right) + \left(C_A^e+1\right)^2$

Using $C_A^e = -\frac{1}{2}$, $C_V^e = -0.03$ (from Table 7.83), Equation 7.93 gives

$$\frac{\sigma(\nu_e e \to \nu_e e)}{\sigma(\nu_\mu e \to \nu_\mu e)} = 6.3 \tag{7.95}$$

which is in good agreement with the experimental result:

$$\frac{\sigma(\nu_e e^- \to \nu_e e^-)}{\sigma(\nu_\mu e^- \to \nu_\mu e^-)} = \frac{0.93\times 10^{-43}\,\mathrm{cm}^2\,(\mathrm{E}/10\ \mathrm{MeV})}{0.16\times 10^{-43}\,\mathrm{cm}^2\,(\mathrm{E}/10\ \mathrm{MeV})} \tag{7.96}$$

The contribution of neutral current, together with the interference between the neutral current and the charged current terms, increases the $\nu_e e \to \nu_e e$ cross-section significantly over the $\nu_\mu e^- \to \nu_\mu e^-$ (and $\nu_\tau e^- \to \nu_\tau e^-$) cross-section. This fact has an interesting consequence for neutrino oscillation. When ν is propagating through a dense medium such as the sun and the earth's core, the effective 'index of refraction' for ν_e is different from that of ν_μ and ν_τ. This modifies the 'potential energy' of ν_e relative to ν_μ and ν_τ and effectively changes the mixing angle between ν_e and ν_μ (ν_τ). One obtains

$$\tan^2\theta_M = \frac{\tan^2\theta_\nu}{1-\left(L_\nu/L_e\right)\sec^2\theta_\nu} \tag{7.97}$$

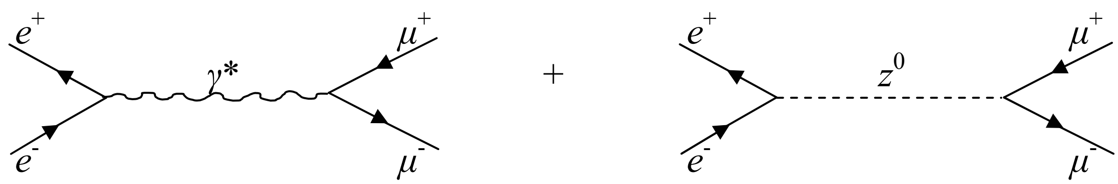
θ_ν is the mixing angle in vacuum

$$L_\nu = \frac{4\pi E_\nu}{\Delta m^2} \qquad L_e = \frac{\sqrt{2}\pi}{Gn_e} \tag{7.97}$$

Δm^2 is the mass difference squared, n_e is the electron density. Equation 7.97 shows that the mixing angle θ_ν can be significantly amplified in matter (when $L_\nu/L_e \sec^2\theta_\nu \to 1$). This is the MSW effect.

The existence of the neutral current also leads to some parity violating γ^* - z^0 interference effects. We consider two examples:

First consider the $e^+e^- \to \mu^+\mu^-$ reaction. The contributing diagrams are

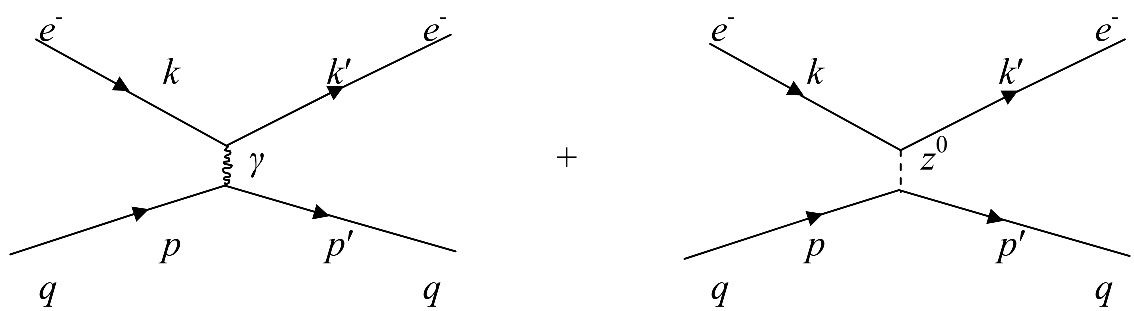


For a pure EM interaction, the angular distribution is 1 + $\cos^2\theta$ c.m. This can be understood from the consideration that the helicities of the e^+, e^-, μ^+, μ^- have the following four possible combinations (and angular distributions).

$$
\begin{array}{ll}
e_R^+e_L^- \to \mu_R^+\mu_L^- & (1+\cos\theta)^2 \\
e_R^+e_L^- \to \mu_L^+\mu_R^- & (1-\cos\theta)^2 \\
e_L^+e_R^- \to \mu_L^+\mu_R^- & (1+\cos\theta)^2 \\
e_L^+e_R^- \to \mu_R^+\mu_L^- & (1-\cos\theta)^2
\end{array}
$$

With a pure vector coupling, these four processes have equal probability, and the angular distribution is ~ 1 + $\cos^2\theta$ (since the terms linear in $\cos\theta$ cancel).

When z^0 term is included, the four processes no longer have equal weighting. Hence the angular distribution is now given by $d\sigma/d\Omega \sim 1 + a\cos\theta + b\cos^2\theta$, with $a \neq 0$. This Forward-Backward asymmetry is observed experimentally, and leads to a determination of $\sin\theta_w$, the weak coupling angle. Another example is the parity violation, observed in e^-N deep-inelastic scattering. The $eq \to eq$ scattering has two terms:



$$M = M_\gamma + M_{z^0}$$
$$M_\gamma = -Q_q e^2 \overline{u}(k')\gamma^\mu u(k)\frac{1}{q^2}\overline{u}(p')\gamma_\mu u(p)$$
$$M_{z^0} = -\frac{g^2}{4\cos^2\theta w}\left[\overline{u}(k')\gamma^\mu\left(C_V^e - C_A^e\gamma^5\right)u(k)\left(\frac{g_{\mu\nu} - q_\mu q_\nu / M_z^2}{q^2 - M_z^2}\right)\right.$$
$$\left.\overline{u}(p')\gamma^\nu\left(C_V^q - C_A^q\gamma^5\right)u(p)\right]$$

For $q^2 \ll M_z^2$, M_{z^0} becomes

$$M_{z^0} = -\frac{g^2}{4\cos^2\theta w M_z^2}\left[\overline{u}(k')\gamma^\mu\left(C_V^e - C_A^e\gamma^5\right)u(k)\overline{u}(p')\gamma_\mu\left(C_V^q - C_A^q\gamma^5\right)u(p)\right]$$

but $$\frac{G}{\sqrt{2}} = \frac{g^2}{8M_z^2\cos^2\theta w}$$ (Equation 13.35 of H & M)

Therefore,

$$M_{z^0} = \sqrt{2}G\left[\overline{u}(k')\gamma^\mu\left(C_V^e - C_A^e\gamma^5\right)u(k)\right]\left[\overline{u}(p')\gamma_\mu\left(C_V^q - C_A^q\gamma^5\right)u(p)\right]$$
$$= \sqrt{2}G\left[\overline{u}(k')\gamma^\mu C_L^e\frac{1}{2}\left(1-\gamma^5\right)u(k) + \overline{u}(k')\gamma^\mu C_R^e\frac{1}{2}\left(1+\gamma^5\right)u(k)\right]$$
$$\left[\overline{u}(p')\gamma_\mu C_L^q\frac{1}{2}\left(1-\gamma^5\right)u(p) + \overline{u}(p')\gamma_\mu C_R^q\frac{1}{2}\left(1+\gamma^5\right)u(p)\right]$$

where

$$C_L^e = C_V^e + C_A^e \qquad C_R^e = C_V^e - C_A^e$$
$$C_L^q = C_V^q + C_A^q \qquad C_R^q = C_V^q - C_A^q$$

$$M_{z^0} = \sqrt{2}G\Big[\bar{u}(k')\gamma^\mu C_L^e \frac{1}{2}(1-\gamma^5)u(k)\bar{u}(p')\gamma_\mu C_L^q \frac{1}{2}(1-\gamma^5)u(p)$$
$$+\bar{u}(k')\gamma^\mu C_L^e \frac{1}{2}(1-\gamma^5)u(k)\bar{u}(p')\gamma_\mu C_R^q \frac{1}{2}(1+\gamma^5)u(p)$$
$$+\bar{u}(k')\gamma^\mu C_R^e \frac{1}{2}(1+\gamma^5)u(k)\bar{u}(p')\gamma_\mu C_L^q \frac{1}{2}(1-\gamma^5)u(p)$$
$$+\bar{u}(k')\gamma^\mu C_R^e \frac{1}{2}(1+\gamma^5)u(k)\bar{u}(p')\gamma_\mu C_R^q \frac{1}{2}(1+\gamma^5)u(p)\Big]$$

Since $$2\gamma^\mu = \gamma^\mu(1+\gamma^5) + \gamma^\mu(1-\gamma^5)$$

we have

$$M_\gamma = -\frac{Q_q e^2}{q^2}\Big[\bar{u}(k')\gamma^\mu \frac{1}{2}(1-\gamma^5)u(k)\bar{u}(p')\gamma_\mu \frac{1}{2}(1-\gamma^5)u(p)$$
$$+\bar{u}(k')\gamma^\mu \frac{1}{2}(1-\gamma^5)u(k)\bar{u}(p')\gamma_\mu \frac{1}{2}(1+\gamma^5)u(p)$$
$$+\bar{u}(k')\gamma^\mu \frac{1}{2}(1+\gamma^5)u(k)\bar{u}(p')\gamma_\mu \frac{1}{2}(1-\gamma^5)u(p)$$
$$+\bar{u}(k')\gamma^\mu \frac{1}{2}(1+\gamma^5)u(k)\bar{u}(p')\gamma_\mu \frac{1}{2}(1+\gamma^5)u(p)\Big]$$

where

$$r = -\frac{\sqrt{2}Gq^2}{e^2}$$

$$\overline{|M|^2} = \frac{1}{4}\sum_{spins}|M|^2$$

Only the ‘diagonal’ terms contribute to $|M|^2$, since the non-diagonal terms all contain factor of $(1-\gamma^5)(1+\gamma^5) = 0$. Therefore, only four terms remain when one evaluates $\overline{|M|^2}$.

$$|M|^2 = |M|^2_{LL\to LL} + |M|^2_{LR\to LR} + |M|^2_{RL\to RL} + |M|^2_{RR\to RR}$$

First consider $LL \to LL$

$$|M|^2_{LL\to LL} = \frac{1}{4}\frac{e^4}{q^4}\left(Q_q + r\ C_L^e C_L^q\right)^2$$

$$\sum_{spin}\left[\bar{u}(k')\gamma^\mu \frac{1}{2}\left(1-\gamma^5\right)u(k)\bar{u}(k)\gamma^\nu \frac{1}{2}\left(1-\gamma^5\right)u(k')\right]$$

$$\left[\bar{u}(p')\gamma_\mu \frac{1}{2}\left(1-\gamma^5\right)u(p)\bar{u}(p)\gamma_\nu \frac{1}{2}\left(1-\gamma^5\right)u(p')\right]$$

$$= \frac{1}{4}\frac{e^4}{q^4}\left(Q_q + rC_L^e C_L^q\right)^2 \frac{1}{16} Tr\left(\gamma^\mu\left(1-\gamma^5\right)\not{k}\gamma^\nu\left(1-\gamma^5\right)\not{k}'\right)$$

$$Tr\left(\gamma_\mu\left(1-\gamma^5\right)\not{p}\gamma_\nu\left(1-\gamma^5\right)\not{p}'\right)$$

$$= \frac{4e^4}{q^4}\left(Q_q + rC_L^e C_L^q\right)^2 (k\bullet p)(k'\bullet p')$$

(using Equation 12.29 of H & M)

but $$(k\bullet p)(k'\bullet p') = s^2/4$$

Therefore, $$\left(\frac{d\sigma}{d\Omega}\right)_{LL\to LL} = \frac{1}{64\pi^2 s}\frac{e^4}{q^4}\left(Q_q + r\ C_L^e C_L^q\right)^2 s^2$$

using $$q^4 = t^2 = \frac{s^2}{4}\left(1-\cos\theta\right)^2$$

$$1-y = \frac{1}{2}\left(1+\cos\theta\right) \qquad y = \frac{1}{2}\left(1-\cos\theta\right)$$

$$\frac{d\sigma}{dy} = \frac{d\sigma}{d\Omega}4\pi \qquad \alpha = \frac{e^2}{4\pi}$$

We have

$$\left(\frac{d\sigma}{dy}\right)_{LL\to LL} = \frac{\pi\alpha^2}{sy^2}\left(Q_q + r\ C_L^e C_L^q\right)^2 \tag{7.99}$$

Similarly

$$\left(\frac{d\sigma}{dy}\right)_{RR\to RR} = \frac{\pi\alpha^2}{sy^2}\left(Q_q + r\ C_R^e C_R^q\right)^2 \tag{7.100}$$

Now, we consider $LR \to LR$

$$\begin{aligned}
|M|^2_{LR\to LR} &= \frac{1}{4}\frac{e^4}{q^4}\left(Q_q + r\ C_L^e C_R^q\right)^2 \times \\
&\quad \sum_{spin}\left[\bar{u}(k')\gamma^\mu \frac{1}{2}\left(1-\gamma^5\right)u(k)\bar{u}(k)\gamma^\nu \frac{1}{2}\left(1-\gamma^5\right)u(k')\right] \\
&\quad \left[\bar{u}(p')\gamma_\mu \frac{1}{2}\left(1+\gamma^5\right)u(p)\bar{u}(p)\gamma_\nu \frac{1}{2}\left(1+\gamma^5\right)u(p')\right] \\
&= 4\frac{e^4}{q^4}\left(Q_q + r\ C_L^e C_R^q\right)^2 (k\bullet p')(k'\bullet p)
\end{aligned}$$

but $$(k\bullet p')(k'\bullet p) = \frac{u^2}{4} = \frac{1}{4}s^2\left(\frac{1+\cos\theta}{2}\right)^2 = \frac{1}{4}s^2(1-y)^2$$

Therefore $$\left(\frac{d\sigma}{dy}\right)_{LR\to LR} = \frac{\pi\alpha^2}{sy^2}\left(Q_q + r\ C_L^e C_R^q\right)^2 (1-y)^2 \tag{7.101}$$

Similarly $$\left(\frac{d\sigma}{dy}\right)_{RL\to RL} = \frac{\pi\alpha^2}{sy^2}\left(Q_q + r\ C_R^e C_L^q\right)^2 (1-y)^2 \tag{7.102}$$

(Note that Equation 13.76 of H & M is incorrect. It misses the y^2 term. Fortunately, the final asymmetry is not affected by this error.)

For an isoscalar target with $u = d$, and ignoring the antiquark contribution,

$$A=\frac{\left(\frac{d\sigma}{dy}\right)_{RR}^{u}+\left(\frac{d\sigma}{dy}\right)_{RR}^{d}+\left(\frac{d\sigma}{dy}\right)_{RL}^{u}+\left(\frac{d\sigma}{dy}\right)_{RL}^{d}-\left(\frac{d\sigma}{dy}\right)_{LR}^{u}-\left(\frac{d\sigma}{dy}\right)_{LR}^{d}-\left(\frac{d\sigma}{dy}\right)_{LL}^{u}-\left(\frac{d\sigma}{dy}\right)_{LL}^{d}}{\left(\frac{d\sigma}{dy}\right)_{RR}^{u}+\left(\frac{d\sigma}{dy}\right)_{RR}^{d}+\left(\frac{d\sigma}{dy}\right)_{RL}^{u}+\left(\frac{d\sigma}{dy}\right)_{RL}^{d}+\left(\frac{d\sigma}{dy}\right)_{LR}^{u}+\left(\frac{d\sigma}{dy}\right)_{LR}^{d}+\left(\frac{d\sigma}{dy}\right)_{LL}^{u}-\left(\frac{d\sigma}{dy}\right)_{LL}^{d}}$$

Using Equations (7.99) – (7.102), we obtain for the numerator of A (ignoring terms quadratic in r):

$$\begin{aligned}A=\frac{\pi\alpha^2}{sy^2}\Bigg[&\left(\frac{4}{9}+2r\left(\frac{2}{3}\right)C_R^eC_R^u+\frac{1}{9}-2r\left(\frac{1}{3}\right)C_R^eC_R^d\right)\\&+\left(\frac{4}{9}+2r\left(\frac{2}{3}\right)C_R^eC_L^u+\frac{1}{9}-2r\left(\frac{1}{3}\right)C_R^eC_L^d\right)\left(1-y^2\right)\\&-\left(\frac{4}{9}+2r\left(\frac{2}{3}\right)C_L^eC_L^u+\frac{1}{9}-2r\left(\frac{1}{3}\right)C_L^eC_L^d\right)\\&-\left(\frac{4}{9}+2r\left(\frac{2}{3}\right)C_L^eC_R^u+\frac{1}{9}-2r\left(\frac{1}{3}\right)C_L^eC_R^d\right)\left(1-y^2\right)\Bigg]\end{aligned}$$

$$\begin{aligned}=\frac{\pi\alpha^2}{sy^2}\left(\frac{2r}{3}\right)\Big\{&\left[2C_R^eC_R^u-C_R^eC_R^d-2C_L^eC_L^u+C_L^eC_L^d\right]\\&+\left(1-y^2\right)\left[2C_R^2C_L^u-C_R^eC_L^d-2C_L^eC_R^u+C_L^eC_R^d\right]\Big\}\end{aligned}$$

$$\begin{aligned}=\frac{\pi\alpha^2}{sy^2}\left(\frac{2r}{3}\right)\Big\{&\left[2C_V^e\left(-2C_A^u+C_A^d\right)+2C_A^e\left(-2C_V^uC_V^d\right)\right]\\&+\left(1-y^2\right)\left[2C_v^e\left(2C_A^u-C_A^d\right)+2C_A^e\left(-2C_V^u+C_V^d\right)\right]\Big\}\end{aligned}$$

where we use $\qquad C_R = C_V - C_A \qquad C_L = C_V + C_A$

with the definition of

$$a_1=C_A^e\left(2C_v^u-C_v^d\right)$$
$$a_2=C_v^e\left(2C_A^u-C_A^d\right)$$

then, we have numerator of

$$A=\frac{\pi\alpha^2}{sy^2}\left(\frac{2r}{3}\right)\left[-2a_1-2a_2+\left(2a_2-2a_1\right)\left(1-y\right)^2\right] \quad (7.103)$$

The denominator of A is simply

$$A=\frac{\pi\alpha^2}{sy^2}\left(\frac{10}{9}\right)\left[1+\left(1-y\right)^2\right] \quad (7.104)$$

where we ignore terms linear or quadratic in r. This is justified since the numerator of A is linear in r and any term linear in r in the denominator only contribute to r^2 in A and can be ignored.

From Equations (7.103) and (7.104), we obtain (recall that $r=\frac{-\sqrt{2}Gq^2}{e^2}$)

$$A=\frac{6}{5}\left(\frac{\sqrt{2}Gq^2}{e^2}\right)\left(a_1+a_2\frac{1-\left(1-y\right)^2}{1+\left(1-y\right)^2}\right)$$

Finally,

$$\begin{aligned}a_1&=C_A^e\left(2C_V^u-C_V^d\right)\\&=\left(-\frac{1}{2}\right)\left[(2)\left(\frac{1}{2}-\frac{4}{3}\sin^2\theta_w\right)-\left(-\frac{1}{2}+\frac{2}{3}\sin^2\theta_w\right)\right]\\&=-\frac{3}{4}\left(1-\frac{20}{9}\sin^2\theta_w\right)\end{aligned}$$

Similarly,

$$\begin{aligned}a_2&=C_V^e\left(2C_A^u-C_A^d\right)\\&=\left(-\frac{1}{2}+2\sin^2\theta_w\right)\left(2\left(\frac{1}{2}\right)-\left(-\frac{1}{2}\right)\right)\\&=-\frac{3}{4}\left(1-4\sin^2\theta_w\right)\end{aligned}$$

The muon decay, $\mu^- \to e^- + \overline{\nu}_e + \nu_\mu$, is an important example of purely leptonic decay. The invariant matrix element is given as

$$M = \frac{G}{\sqrt{2}}\left[\overline{u}(k)\gamma^\mu\left(1-\gamma^5\right)u(p)\right]\left[\overline{u}(p')\gamma_\mu\left(1-\gamma^5\right)v(k')\right]$$

where the 4-vectors correspond to

$$\mu^-(p) \to e(p') + \overline{\nu}_e(k') + \nu_\mu(k)$$

It is straight forward to obtain

$$\overline{|M|^2} = 64G^2(k\bullet p')(k'\bullet p)$$

The decay width is given by

$$d\Gamma = \frac{1}{2E}\overline{|M|^2}\,dQ$$

where the Lorentz Invariant Phase Space dQ is

$$dQ = \frac{d^3p'}{(2\pi)^3 2E'}\,\frac{d^3k}{(2\pi)^3 2w}\,\frac{d^3k'}{(2\pi)^3 2w'}\,(2\pi)^4\delta^4(p-p'-k-k')$$

Integrating over the Delta function (see H & M 11.5 for details) one obtains

$$\frac{d\Gamma}{dE'} = \frac{G^2}{12\pi^3}m^2E'^2\left(3-\frac{4E'}{m}\right)$$

and the decay width

$$\Gamma = \frac{1}{\tau} = \int_0^{m/2} dE'\left(\frac{d\Gamma}{dE'}\right) = \frac{G^2m^5}{192\pi^3}$$

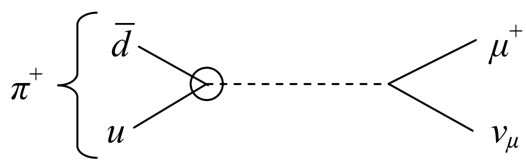
The decay width is proportional to the fifth power of the mass of the decaying particle.

Similar results can be obtained for other decays such as

$$\tau^- \to e^- \overline{\nu}_e \nu_\tau$$
$$b \to c\overline{\nu}\ell^-$$
$$t \to b e^+ \nu_e$$

We consider next the $\pi^+ \to \mu^+ + \nu_\mu$ decay, which is an example of semileptonic decay.

As in cases which involve hadrons, we need to parameterize the hadronic weak current based on general principles.



$$\pi^+(q) \to \mu^+(p) + \nu_\mu(k)$$

The leptonic current is

$$\overline{u}(p)\gamma_\mu\left(1-\gamma^5\right)v(k)$$

The hadronic current can only be a combination of V and A (in order to make the invariant amplitude a scalar or pseudoscalar).

$$M = \frac{G}{\sqrt{2}}\left(f_\pi q^\mu\right)\left[\overline{u}(p)\gamma_\mu\left(1-\gamma^5\right)v(k)\right]$$

The only V or A which can be constructed from a single spin-0 object (like π) is q^μ, and f_π represents the pion structure factor.

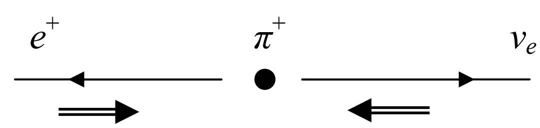
$$\overline{|M|^2} = 4G^2 f_\pi^2 m_\mu^2 \left(p \bullet k\right)$$

Note that the mass^2 of muon enters. For $\pi^+ \to e^+ + \nu_e$ decay, the m_e^2 term greatly reduces the decay probability.

$$P = \frac{1}{\tau} = \frac{G^2}{8\pi} f_\pi^2 m_\pi m_\mu^2 \left(1 - \frac{m_\mu^2}{m_\pi^2}\right)^2$$

and

$$\frac{P\left(\pi^+ \to e^+\nu_e\right)}{P\left(\pi^+ \to \mu^+\nu_\mu\right)} = \left(\frac{m_e}{m_\mu}\right)^2 \left(\frac{m_\pi^2 - m_e^2}{m_\pi^2 - m_\mu^2}\right)^2 = 1.2 \times 10^{-4}$$



Since π^+ has spin-0, and ν_e has negative helicity, e^+ is required to be left-handed in order to conserve angular momentum. e^+ is therefore in the wrong helicity state, which inhibits the decay probability.

Similar suppression is observed for k decays:

$$\frac{\Gamma\left(k^- \to e^-\overline{\nu}_e\right)}{\Gamma\left(k^- \to \mu^-\overline{\nu}_\mu\right)} = 2.1 \times 10^{-5}$$

One can also compare the $k^- \to \mu^-\overline{\nu}_\mu$ with $\pi^- \to \mu^-\overline{\nu}_\mu$:

$$R = \frac{\Gamma\left(k^- \to \mu^-\nu_\mu\right)}{\Gamma\left(\pi^- \to \mu^-\nu_\mu\right)} = \frac{\sin^2\theta_c}{\cos^2\theta_c} \frac{f_k^2 m_k}{f_\pi^2 m_\pi} \frac{\left[1 - \left({m_\mu}/{m_k}\right)^2\right]^2}{\left[1 - \left({m_\mu}/{m_\pi}\right)^2\right]^2}$$

From the experimental value of R, one determines

$$f_k / f_\pi = 1.28$$

Another decay related to $\pi \to \mu\nu$ is

$$\tau^- \to \pi^-\nu_\tau$$

or

$$\tau^- \to k^- \nu_\tau$$

The decay width is given as

$$\Gamma\left(\tau^- \to \pi^- \nu_\tau\right) = \frac{G^2 f_\pi^2 m_\tau^3}{16\pi}\left(1 - \frac{m_\pi^2}{m_\tau^2}\right)^2$$

Note that there is no helicity suppression in this decay, and

$$\Gamma\left(\tau^- \to \pi^- \nu_\tau\right) > \Gamma\left(\tau^- \to k^- \tau_\tau\right)$$

from phase-space consideration (and the cabbibo angle θ_c and f_k / f_π considerations).

Another example of semi-leptonic weak decay is

$$\pi^+ \to \pi^0 \ell^+ \nu$$

or $\qquad\qquad\qquad\qquad\qquad\qquad \left(M_a \to M_b \ell \nu\right)$

$$k^+ \to \pi^0 \ell^+ \nu$$

Now, we have two spin-0 hadrons. Therefore, we have two vectors, k_a and k_b, available for constructing the hadronic current:

$$\left\langle M_b \middle| J^\alpha \middle| M_a \right\rangle = N\left(f_a k_a^\alpha + f_b k_b^\alpha\right)$$

One can also extend this to baryonic semileptonic decay

$$B \to B' \ell \nu$$

We now have two spin-½ Dirac particles. One can form vector and axial-vector hadronic currents of various forms vector:

$$\bar{u}(B')\gamma^{\alpha}u(B)$$
$$\bar{u}(B')\sigma^{\alpha\nu}q_{\nu}u(B)$$
$$\bar{u}(B')q^{\alpha}u(B)$$

axial-vector:

$$\bar{u}(B')\gamma^{\alpha}\gamma^{5}u(B)$$
$$\bar{u}(B')\sigma^{\alpha\nu}q_{\nu}\gamma^{5}u(B)$$
$$\bar{u}(B')q^{\alpha}\gamma^{5}u(B)$$

$$\begin{aligned}\langle B'|J^{\alpha}|B\rangle = N\bar{u}(B')\Big[& f_1(q^2)\gamma^{\alpha} + if_2(q^2)\sigma^{\alpha\nu}q_{\nu} + f_3(q^2)q^{\alpha} \\ & - g_1(q^2)\gamma^{\alpha}\gamma^{5} - ig_2(q^2)\sigma^{\alpha\nu}q_{\nu}\gamma^{5} \\ & - g_3(q^2)\gamma^{5}q^{\alpha}\Big]u(B)\end{aligned}$$

This concludes our discussion on weak interaction.