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Dec. 9, 2013 Lecture

~~the~~ Gauge invariance, Electroweak unification
Standard Model

* Gauge invariance

$$\mathcal{L}_0^{Dirac} = \bar{\psi}(i\gamma^\mu \partial_\mu - m)\psi$$

Euler - Lagrange Eq.

$$\frac{\partial}{\partial x_\mu}\left(\frac{\partial \mathcal{L}_0}{\partial(\partial\psi/\partial x_\mu)}\right) - \frac{\partial \mathcal{L}_0}{\partial \psi} = 0$$

$\Rightarrow$ Dirac Eq. $\quad (i\gamma^\mu\partial_\mu - m)\psi = 0$

$$\mathcal{L}_0^{K\text{-}G} = \frac{1}{2}(\partial_\mu\phi)(\partial^\mu\phi) - \frac{1}{2}m^2\phi^2$$

$\Rightarrow$ Klein - Gordon eq. $\quad (\partial_\mu\partial^\mu + m^2)\phi = 0$

$$\mathcal{L}_0^{photon} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - j^\mu A_\mu$$

$$F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu$$

$\Rightarrow \partial_\mu F^{\mu\nu} = j^\nu \quad$ [Maxwell Eq.]

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Consider

$$\mathcal{L}_0^{Dirac} = \bar{\psi}\,(i\gamma^\mu \partial_\mu - m)\,\psi$$

1) Global $U(1)$ transformation

$$\psi'(x) = e^{-iqw}\,\psi(x)$$

(q is fixed, w is the parameter specifying the transformation)

$$\bar{\psi}'(x) = e^{iqw}\,\bar{\psi}(x)$$

$$\partial_\mu \psi(x) \to \partial_\mu \psi'(x) = e^{-iqw}\partial_\mu \psi(x)$$

(field gradient transforms the same as the field)

$$\mathcal{L}_0^{Dirac\,\prime} = \mathcal{L}_0^{Dirac}$$

(invariant under the global $U(1)$ transformation)

2) Local $U(1)$ transformation

$$\psi'(x) = e^{-iq\,w(x)}\,\psi(x)$$

transformation depends on space-time

$$\partial_\mu \psi'(x) = e^{-iqw}\left(\partial_\mu \psi(x) - iq\,(\partial_\mu w(x))\,\psi(x)\right)$$

Field gradient transforms differently from the field.

and

$$\mathcal{L}_0' = \mathcal{L}_0 + q\bar{\psi}\gamma^\mu\psi\,\partial_\mu w = \mathcal{L}_0 + j^\mu \partial_\mu w$$

$\mathcal{L}_0$ is no longer invariant under this transformation

This can be remedied by introducing a vector field A_μ which couples to the current j^μ

$$\boxed{\mathcal{L}_1 = \mathcal{L}_0 - j^\mu A_\mu}$$

add another term (interaction term) to the Lagrangian

$$\mathcal{L}_1' = \mathcal{L}_0' - j'^\mu A_\mu' = \mathcal{L}_0 + j^\mu \partial_\mu w - j'^\mu A_\mu'$$

$$(j_\mu' = q\bar{\psi}'\gamma_\mu\psi' = j_\mu)$$

$$A_\mu' = A_\mu + \partial_\mu w$$

$$\Rightarrow \mathcal{L}_1' = \mathcal{L}_0 - j^\mu A_\mu = \mathcal{L}_1, \quad \text{invariant under local } U(1)$$

A_μ is the 'gauge field'

$$\mathcal{L}_1 = \bar{\psi}(i\gamma^\mu\partial_\mu - m)\psi - q\bar{\psi}\gamma^\mu\psi A_\mu$$
$$= \bar{\psi}(i\gamma^\mu D_\mu - m)\psi$$

$D_\mu = \partial_\mu + iqA_\mu$ is the covariant derivative

$$D_\mu\psi(x) \rightarrow D_\mu'\psi'(x) = e^{iqw(x)} D_\mu\psi(x)$$

The field gradient now transforms the same way as the field:

√

One can add other terms built up from the gauge field A_μ and it's derivative which are also gauge invariant

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$$ is also invariant

under $A_\mu \to A_\mu + \partial_\mu w$ transformation

$$\mathcal{L}_1 = \bar{\psi}(i\gamma^\mu D_\mu - m)\psi - \frac{1}{4} \underbrace{F_{\mu\nu} F^{\mu\nu}}_{\text{Lorentz scalar}}$$

a mass term $\frac{1}{2} m^2 A_\mu A^\mu$ is not allowed, since it is not invariant under $A_\mu \to A_\mu + \partial_\mu w$
(not gauge invariant) (gauge field is massless)

(Abelian → Non-abelian)

3) Now consider $SU(n)$ $n > 1$

we have n different Dirac fields ψ^a, $a = 1 \cdots n$

(a denotes internal degree of freedom, such as isospin or color)

$$\psi^a \to \psi^{a'} = U^a{}_b \psi^b$$ U is $n \times n$ unitary matrix

$$U = \exp(-ig\, w_i T_i), \quad i = 1 \to N$$

T_i : generators $\quad N = n^2 - 1$

$$[T_i, T_j] = i f_{ijk} T_k$$ commutation relations

$n = 2 \Rightarrow N = 3$, Pauli-matrices $\quad n = 3 \Rightarrow N = 8$ Gell-Mann

Free-field Lagrangian density

$$\mathcal{L}_0 = \sum_{a=1}^{n} \bar{\psi}_a \left(i \gamma^\mu \partial_\mu - m \right) \psi^a$$

a refers to isospin ($SU(2)$) or color ($SU(3)$) indices ($n = 3$, $\overset{SU(3)}{\psi_R, \psi_B, \psi_G}$; $n=2$, $\overset{SU(2)}{\psi_{I_z=+\frac{1}{2}}, \psi_{I_z=-\frac{1}{2}}}$)

$\mathcal{L}_0$ is invariant under global $SU(n)$ transformation with

$$U = \exp\left(-i g w_i T_i \right)$$

4) If U depends on x then

$$\mathcal{L}_0 \to \mathcal{L}_0' = \mathcal{L}_0 + \bar{\psi} i \gamma^\mu \left(U^\dagger \partial_\mu U \right) \psi$$

Again, the field and the field gradien transform differently.

$$\psi \to \psi' = U \psi$$

$$\partial_\mu \psi \to \partial_\mu \psi' = U \left(\partial_\mu \psi \right) + \left(\partial_\mu U \right) \psi$$

Introduce A_μ ($n \times n$ matrix) whose elements are gauge fields

$$A_\mu = A_{j\mu} T_j$$

$$A_\mu \to A_\mu' = \frac{i}{g} \left(\partial_\mu U \right) U^\dagger + U A_\mu U^\dagger$$

$$\mathcal{L}_1 = \mathcal{L}_0 - g \bar{\psi} \gamma^\mu A_\mu \psi$$

(6)

$$[T_i, T_j] = i f_{ijk} T_k$$

$$A_{i\mu} \to A_{i\mu}' = A_{i\mu} + \partial_\mu w_i - g f_{ijk} A_{j\mu} w_k$$

$i = 1, \cdots n^2-1$ local tran. non-abelian

SU(2) : 3 gauge fields, SU(3) : 8 gauge fields

$$D_\mu = \partial_\mu + ig A_\mu \quad (\text{where } A_\mu = A_{j\mu} T_j)$$

$$\mathcal{L}_1 = \bar{\psi}(i\gamma^\mu D_\mu - m)\psi$$

Have to add The gauge field term to the Lagrangian

$$\mathcal{L}_G = -\frac{1}{4} F^i_{\mu\nu} F_i^{\mu\nu}$$

$$F^i_{\mu\nu} = \partial_\mu A^i_\nu - \partial_\nu A^i_\mu - g f_{ijk} A^j_\mu A^k_\nu$$

(For $\mathcal{L}_G$ to be invariant under local SU(n) transformation)

since $A^i_\mu \to A^{i\prime}_\mu = A^i_\mu + \partial_\mu w^i - g f_{ijk} A_{j\mu} w_k$

$$\mathcal{L} = \mathcal{L}_1 + \mathcal{L}_G$$

$$\mathcal{L}_1 = \bar{\psi}(i\gamma^\mu D_\mu - m)\psi$$
$$= \bar{\psi}(i\gamma^\mu \partial_\mu - m)\psi - g A_{i\mu} \bar{\psi}\gamma^\mu T_i \psi$$

$$\mathcal{L}_G = -\frac{1}{4} B^i_{\mu\nu} B_i^{\mu\nu} + \frac{g}{2} f_{ijk} B^i_{\mu\nu} A^\mu_j A^\nu_k - \frac{g^2}{4} f_{ijk} f_{ilm} A_{j\mu} A_{k\nu} A^\mu_l A^\nu_m$$

$(B^i_{\mu\nu} = \partial_\mu A^i_\nu - \partial_\nu A^i_\mu)$

$$QCD \longleftrightarrow SU(3) \qquad \begin{pmatrix} \psi^i_R \\ \psi^i_B \\ \psi^i_G \end{pmatrix}$$

$$\mathcal{L}_{QCD} = \sum_{A=1}^{N_f} \bar{\psi}^A (i \gamma^\mu D_\mu - m_A) \psi^A - \frac{1}{4} F^i_{\mu\nu} F_i^{\mu\nu}$$

$N_f = 6$, $\quad i = 1 \to \underset{\substack{\uparrow \\ n^2-1 \ (n=3)}}{8}$. 8 gluon field

Flavor-independent, except mass term m_A

Contains $N_f + 1$ parameters

$$D_\mu \psi^A_a = \partial_\mu \psi^A_a + \frac{i}{2} g_s G_{i\mu} (\lambda_i)_{ab} \psi^A_b$$

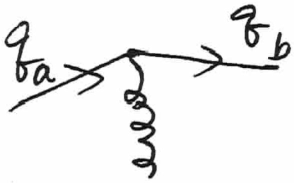
$$F^i_{\mu\nu} = \partial_\mu G^i_\nu - \partial_\nu G^i_\mu - g_s f_{ijk} G^j_\mu G^k_\nu$$

From $\bar{\psi}(i \gamma^\mu \partial_\mu \psi)$ in $\mathcal{L}$

and $\partial_\mu \to D_\mu$

Quark-gluon coupling term

$$- g_s \bar{\psi}(x) \gamma^\mu \frac{\lambda_i}{2} \psi(x) G_{i\mu}(x)$$

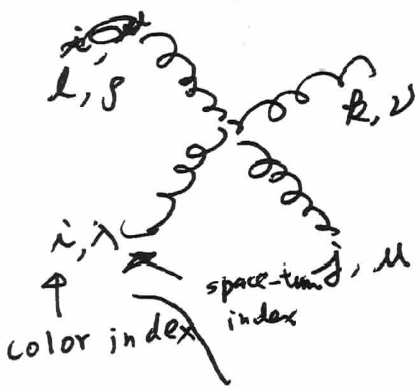


$$- i g_s \gamma_\mu \left(\frac{\lambda_i}{2} \right)_{ba}$$

$(-i q \gamma_\mu)$ ← for electromagnetic

$$\mathcal{L}_Q = -\frac{1}{4} F^i_{\mu\nu} F_i^{\mu\nu} = -\frac{1}{4} G^i_{\mu\nu} G_i^{\mu\nu} + \frac{g_s}{2} f_{ijk} G^i_{\mu\nu} G^\mu_j G^\nu_k$$
$$- \frac{g_s^2}{4} f_{ijk} f_{ilm} G_{j\mu} G_{k\nu} G^\mu_l G^\nu_m$$

$$G^i_{\mu\nu} = \partial_\mu G^i_\nu - \partial_\nu G^i_\mu$$



$$-ig_s^2\{f_{ijm}f_{klm}(g_{\lambda\nu}g_{\mu\rho}-g_{\mu\nu}g_{\lambda\rho}) + f_{ikm}f_{jlm}(g_{\lambda\mu}g_{\nu\rho}-g_{\mu\nu}g_{\lambda\rho}) + f_{kjm}f_{ilm}(g_{\lambda\nu}g_{\mu\rho}-g_{\lambda\mu}g_{\nu\rho})\}$$

How about extending this to weak interaction?

SU(2) $\quad W^0 \begin{pmatrix} u \\ d \end{pmatrix} W^\pm \quad \begin{pmatrix} \nu_e \\ e^- \end{pmatrix} \quad \begin{pmatrix} n^2-1 = 3 \\ \text{gauge bosons} \end{pmatrix}$

$\begin{pmatrix} u_R \\ u_B \\ u_G \end{pmatrix}$

However, the experimental evidences are $W^\pm$ are massive!

And local gauge invariance is broken by the mass term of the gauge boson!

(If it ~~not a~~ does not have local gauge invariance, then it is not renormalizabl

$\Rightarrow$ need spontaneous symmetry breaking

Higgs mechanism

Let us see how the standard model for Electroweak Interaction is obtained

Consider one lepton family for now, (ν, e)

$$\psi(x) = \psi_L(x) + \psi_R(x)$$

$$\psi_L(x) = a_L\,\psi(x) \qquad a_L = \tfrac{1}{2}(1-\gamma_5)$$

$$\psi_R(x) = a_R\,\psi(x) \qquad a_R = \tfrac{1}{2}(1+\gamma_5)$$

observation of left-handed leptons in weak interactions ~~impl~~ suggests the following lepton fields

$$\psi_L = \begin{pmatrix} \nu_L \\ e_L \end{pmatrix} \qquad \psi_R = (e_R) \qquad \cancel{\nu_R} \text{ does not exist}, \; m(\nu_R)=0$$

$\boxed{\leftarrow \text{member of weak isospin singlet}}$

members of weak isospin doublet

$\psi_R = e_R$ ~~member~~

$\boxed{\text{only can have a scalar term with } \bar{\psi}_L m \psi_L \text{ or } \bar{\psi}_R m \psi_R \text{ both are zero}}$

$$\mathcal{L}_0 = \bar{\psi}_L\, i\gamma^\mu \partial_\mu \psi_L + \bar{\psi}_R\, i\gamma^\mu \partial_\mu \psi_R$$

$$= \bar{e}\, i\gamma^\mu \partial_\mu e + \bar{\nu}_L\, i\gamma^\mu \partial_\mu \nu_L$$

$$e = \begin{pmatrix} e_L \\ e_R \end{pmatrix}$$

Contain both e_L and e_R

(free Lagrangian massless)

$$U = e^{-ig w_i t_i} \qquad \text{for } SU(2) \text{ transformation}$$

$$t_i, \; w_i, \; i = 1, 2, 3$$

$t_{iL} = \frac{1}{2}\tau_i$, when it operates on the ψ_L doublet

$\left(t_{iR} \overset{U=1}{=} 0\right)$, when " ψ_R singlet

The Lagrangian is also invariant under $U(1)$ transformation

$$U(w) = e^{-i f w F}$$

f: coupling constant, F: generator for $U(1)$

What is F? (Is it equal to Q)

$\psi_L = \begin{pmatrix} \nu_L \\ e_L^- \end{pmatrix}$ contains member of different charge

They have the same hypercharge Y where $Y = 2(Q - T_3)$

(More formal derivation by considering the conserved current is $SU_L(2)$ and $U(1)$. They must commute)

Particle	I	T_3	Q	Y
ν_e	$\frac{1}{2}$	$+\frac{1}{2}$	0	-1
e_L	$\frac{1}{2}$	$-\frac{1}{2}$	-1	-1
e_R	0	0	-1	-2

Now, we ~~move to~~ consider local $SU_L(2) \times U(1)$ gauge invariance

$$\mathcal{L}_0 = \bar{\psi}_L i \gamma^\mu D^L_\mu \psi_L + \bar{\psi}_R i \gamma^\mu D^R_\mu \psi_R$$

(replacing ∂_μ by covariant derivative D_μ)

(left couples to $SU(2)$ and $U(1)$ gauge bosons)

$$D^L_\mu \psi_L = \left(\partial_\mu + i g A_{i\mu} \frac{\tau_i}{2} + i g' B_\mu \frac{Y_L}{2} \right) \psi_L$$

(right only couples to $U(1)$)

$$D^R_\mu \psi_R = \left(\partial_\mu + i g' B_\mu \frac{Y_R}{2} \right) \psi_R$$

We now have ~~2 vector~~ 3 gauge fields ($A_{i\mu}$) associated with $SU_L(2)$, and one gauge field for $U_Y(1)$ (~~B_μ~~) (B_μ)

The dynamics of the gauge fields are contained in

$$\mathcal{L}_G = -\frac{1}{4} W^i_{\mu\nu} W_i^{\mu\nu} - \frac{1}{4} B_{\mu\nu} B^{\mu\nu}$$

where

$$W^i_{\mu\nu} = \partial_\mu A^i_\nu - \partial_\nu A^i_\mu - g\, \epsilon_{ijk} A^j_\mu A^k_\nu$$

and

$$B_{\mu\nu} = \partial_\mu B_\nu - \partial_\nu B_\mu$$

$$A'_\mu = A_\mu + \partial_\mu w + i g [A_\mu, w]$$