Chapter 5

Quantum Electrodynamics

1. EM Interactions of spin-0 Particles

We now consider electromagnetic interactions involving spin-0 particles. These particles are assumed to be structureless point-like charged particles. We would like to describe scattering processes such as

$$\begin{aligned} &\pi^+\pi^- \to \pi^+\pi^- \\ &\pi^+\pi^- \to K^+K^- \\ &\pi^+ A \to \pi^+ A \end{aligned} \tag{5.1}$$

or a fictitious ‘spinless’ electron scattering off a spinless electron or muon

$$\begin{aligned} &{}'e^-{}''e^-{}' \to {}'e^-{}'+{}'e^-{}' \\ &e^-\mu^- \to e^-\mu^- \\ &e^-e^+ \to \mu^-\mu^+ \qquad \text{etc.} \end{aligned} \tag{5.2}$$

To study the transition rate and cross section of these processes, we start by considering the lowest order perturbation theory. The transition amplitude T_{fi} is given as

$$T_{fi} = -i\int d^4x\ \phi_f^*(x)\ v(x)\ \phi_i(x) \tag{5.3}$$

For spin-0 particles, the Klein-Gordon equation

$$\left(\partial_\mu \partial^\mu + m^2\right)\ \phi(x) = 0 \tag{5.4}$$

gives the following plane-wave solutions

$$\begin{aligned} \phi_i(x) &= N_i e^{-iP_i \bullet x} \\ \phi_f^*(x) &= N_f e^{iP_f \bullet x} \end{aligned} \tag{5.5}$$

Interaction of a charged particle in an EM potential $A^{\mu}=\left(A^{0},\,\vec{A}\right)$ is obtained by the substitution

$$i\partial^{\mu}\rightarrow i\partial^{\mu}+eA^{\mu} \tag{5.6}$$

The Klein-Gordon equation becomes

$$\left(\partial_{\mu}\partial^{\mu}+V+m^{2}\right)\,\phi\left(x\right)=0 \tag{5.7}$$

where $$V=-ie\left(\partial_{\mu}A^{\mu}+A^{\mu}\partial_{\mu}\right)-e^{2}A^{2}$$

Substituting Equation 5.7 into Equation 5.3 and ignoring the higher order (in e) term $e^{2}A^{2}$, we obtain

$$T_{fi}=-e\int\phi_{f}^{*}\left(A^{\mu}\partial_{\mu}+\partial_{\mu}A^{\mu}\right)\,\phi_{i}\;d^{4}x \tag{5.8}$$

Integration by part changes the second term of Equation 5.8 into

$$\int\phi_{f}^{*}\partial_{\mu}\left(A^{\mu}\phi_{i}\right)\;d^{4}x=-\int\partial_{\mu}\left(\phi_{f}^{*}\right)A^{\mu}\phi_{i}d^{4}x \tag{5.9}$$

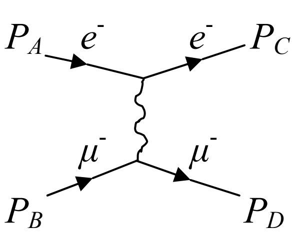
and T_{fi} can be written as

$$T_{fi}=-i\int j_{\mu}^{fi}A^{\mu}d^{4}x \tag{5.10}$$

where the transition current j_{μ}^{fi} between states i and f is

$$j_{\mu}^{fi}=-ie\left[\phi_{f}^{*}\left(\partial_{\mu}\phi_{i}\right)-\left(\partial_{\mu}\phi_{f}^{*}\right)\phi_{i}\right] \tag{5.11}$$

Consider a $2\rightarrow 2$ process such as spin-less $e^{-}\mu^{-}\rightarrow e^{-}\mu^{-}$ scattering



$$j_{\mu}^{fi}=-eN_{A}N_{C}\left(P_{A}+P_{C}\right)_{\mu}e^{i\left(P_{C}-P_{A}\right)\bullet x}$$

or

$$j_{\mu}^{(1)}=-eN_{A}N_{C}\left(P_{A}+P_{C}\right)_{\mu}e^{i\left(P_{C}-P_{A}\right)\bullet x} \tag{5.12}$$

What is the EM potential A^μ generated by μ^-? The Maxwell Equation under the Lorentz condition, $\partial_\mu A^\mu = 0$, becomes

$$\Box^2 A^\mu = j^\mu \tag{5.13}$$

The current j^μ of the muon is analogous to that of the electron, and is given as

$$j^\mu_{(2)} = -eN_B N_D \left(P_B + P_D\right)^\mu e^{i(P_D - P_B)x} \tag{5.14}$$

Keeping in mind that

$$\Box^2 e^{+iq\cdot x} = -q^2 e^{+iq\cdot x} \tag{5.15}$$

The solution to Equation 5.13 can be obtained by inspection:

$$A^\mu = -\frac{1}{q^2} j^\mu_{(2)} \tag{5.16}$$

where

$$q = P_D - P_B$$

The Transition amplitude becomes

$$T_{fi} = -i\int j^{(1)}_\mu(x)\left(-\frac{1}{q^2}\right) j_2^{(\mu)}(x)\, d^4x \tag{5.17}$$

Substituting Equations 5.12 and 5.14 into Equation 5.17 and noting that

$$\int e^{i(P_C - P_A + P_D - P_B)\cdot x} d^4x = (2\pi)^4 \delta^{(4)}\left(P_D + P_C - P_B - P_A\right) \tag{5.18}$$

we obtain

$$T_{fi} = -iN_A N_B N_C N_D = (2\pi)^4 \delta^{(4)}\left(P_D + P_C - P_B - P_A\right) M \tag{5.19}$$

where

$$-iM = \left[ie\left(P_A + P_C\right)^\mu\right]\left(-i\frac{g_{\mu\nu}}{q^2}\right)\left[ie\left(P_B + P_D\right)^\nu\right] \tag{5.20}$$

M is Lorentz invariant and called the ‘invariant amplitude’.

For $A + B \rightarrow C + D$, the transitions per unit time per unit volume is

$$W_{fi} = \frac{|T_{fi}|^2}{T \bullet V} \tag{5.21}$$

It can be shown that the transition time *T* and the volume *V* in Equation 5.21 cancel the delta function in T_{fi} specifically.

$$T_{fi} \ \alpha (2\pi)^4 \delta^4 (P_D + P_C - P_B - P_A) \tag{5.22}$$

and

$$|T_{fi}|^2 \alpha \ (2\pi)^8 \delta^4 (P_D + P_C - P_B - P_A) \ \delta^4 (P_D + P_C - P_B - P_A) \tag{5.23}$$

Consider the 0-th component of the delta function in Equation 5.23:

$$\begin{aligned} &(2\pi)^2 \delta(\mathrm{E}_F - \mathrm{E}_I)\delta(\mathrm{E}_F - \mathrm{E}_I) \\ &= (2\pi) \ \delta(\mathrm{E}_F - \mathrm{E}_I) \int_{-T/2}^{T/2} e^{i(\mathrm{E}_F - \mathrm{E}_I)t} dt \\ &= (2\pi) \ \delta(\mathrm{E}_F - \mathrm{E}_I) \bullet 2 \bullet \frac{\sin\left[\left(T/2\right)(\mathrm{E}_F - \mathrm{E}_I)\right]}{\mathrm{E}_F - \mathrm{E}_I} \end{aligned} \tag{5.24}$$

The first delta function in Equation 5.24 requires

$$\frac{\sin\left[\left(T/2\right)(\mathrm{E}_F - \mathrm{E}_I)\right]}{\mathrm{E}_F - \mathrm{E}_I} = \frac{T}{2} \tag{5.25}$$

and Equation 5.24 becomes

$$(2\pi)^2 \delta(\mathrm{E}_F - \mathrm{E}_I) \ \delta(\mathrm{E}_F - \mathrm{E}_I) = (2\pi) \ \delta(\mathrm{E}_F - \mathrm{E}_I) T \tag{5.26}$$

Similarly, one can show that

$$\begin{aligned} &|T_{fi}|^2 \alpha (2\pi)^4 \delta^4 (P_D + P_C - P_B - P_A)(T)(L_x)(L_y)(L_z) \\ &= (2\pi)^4 \delta^4 (P_D + P_C - P_B - P_A) \bullet T \bullet V \end{aligned} \tag{5.27}$$

To convert W_{fi} into the cross section, $d\sigma$, which characterized the effective area within which the $A + B$ collision can lead to $C + D$, one needs to multiply W_{fi} by the number of final states and divide it by the initial flux:

$$d\sigma = \frac{W_{fi}}{(\text{initial flux})} \times (\text{number of final states}) \quad (5.28)$$

Adopting the so-called covariant normalization for the Klein-Gordon equation

$$\int_V \zeta\, dv = 2\text{E} \qquad N = \frac{1}{\sqrt{v}} \quad (5.29)$$

The initial flux is therefore proportional to the number of beam particles passing through unit area per unit time, $|V_A| 2\text{E}_A / V$, and the number of target particles per unit volume, $2\text{E}_B/\text{V}$.

$$\text{Initial Flux} = \left|\vec{V}_A\right| \frac{2E_A}{V} \frac{2E_B}{V} \quad (5.30)$$

The number of final states in a volume V with momentum within $\alpha^3 P$ is $V\,\alpha^3 P/(2\pi)^3$. Since there are 2E particles in V, we have

$$\text{Number of final states / particle} = \frac{Vd^3P}{(2\pi)^3\, 2\text{E}} \quad (5.31)$$

And the number of available final states for particles C, D scattered into $\alpha^3 P_C$, $\alpha^3 P_D$ is

$$\frac{Vd^3P_C}{(2\pi)^3\, 2\text{E}_C} \; \frac{Vd^3P_D}{(2\pi)^3\, 2\text{E}_D} \quad (5.32)$$

Inserting Equations 5.21, 5.27, 5.19, 5.30, 5.32 into Equation 5.28, we finally obtain

$$d\sigma = \frac{|M|^2}{F} dQ \quad (5.33)$$

where

$$dQ = (2\pi)^4 \delta^{(4)}(P_C + P_D - P_A - P_B) \frac{d^3 P_C}{(2\pi)^3 2E_C} \frac{d^3 P_D}{(2\pi)^3 2E_D} \qquad (5.34)$$

is the Lorentz invariant phase space factor (αLips) and the flux factor F is

$$F = |V_A| \bullet 2E_A \bullet 2E_B \qquad (5.35)$$

in the lab frame.

For a general collinear collision between A and B

$$\begin{aligned} F &= \left|\vec{V}_A - \vec{V}_B\right| \bullet 2E_A \bullet 2E_B = \left|\vec{V}_A\right| + \left|\vec{V}_B\right| \bullet 2E_A \bullet 2E_B \\ &= 4\left(\left|\vec{P}_A\right| E_B + \left|\vec{P}_B\right| E_A\right) \qquad \left(\left|\vec{V}\right| = \frac{\left|\vec{P}\right|}{E}\right) \qquad (5.36) \\ &= 4\left[\left(P_A \bullet P_B\right)^2 - M_A^2 M_B^2\right]^{1/2} \end{aligned}$$

In the center-of-mass frame for the process $A + B \rightarrow C + D$, one can show (Ex. 4.2 of H & M)

$$dQ = \frac{1}{4\pi^2} \frac{P_f}{4\sqrt{S}} d\Omega \qquad (5.37)$$

S is the square of center-of-mass energy

and from 5.36,

$$F = 4P_i\sqrt{S} \qquad (5.38)$$

where P_i, P_f are the initial and final 3-momentum in the C.M. frame.

Equations 5.33, 5.37, 5.38 give the following important expression for the differential cross-section in the C.M. frame:

$$\left(\frac{d\sigma}{d\Omega}\right)_{CM} = \frac{1}{64\pi^2 S} \frac{P_f}{P_i} |M|^2 \qquad (5.39)$$

Note added:

Here is the derivation for Equation 5.37:

$$dQ^{(6)} = (2\pi)^4 \delta^4 (P_3 + P_4 - P_1 - P_2) \frac{d^3P_3\ d^3P_4}{(2\pi)^3 2E_3 (2\pi)^3 2E_4}$$

To evaluate dQ in the C.M. frame $\left(\vec{P}_1 = -\vec{P}_2,\ \vec{P}_3 = -\vec{P}_4\right)$, we first integrate over d^3P_4:

$$dQ^{(3)} = \frac{\delta(E_3 + E_4 - W)}{16\pi^2 E_3 E_4}\ d^3P_3$$

where $$W = E_1 + E_2 = E_3 + E_4 = E_{C.M.} = \sqrt{S}$$

To proceed further, we need to express d^3P_3 in terms of dE_3 and express E_4 in terms of E_3:

$$d^3P_3 = P_3^2 dP_3 d\Omega = P_3 E_3 dE_3 d\Omega$$

since $$P_3 dP_3 = E_3 dE_3 \left(E_3^2 = P_3^2 + M_3^2\right)$$

$$E_3 + E_4 - W = E_3 + \left(E_3^2 - M_3^2 + M_4^2\right)^{1/2} - W$$

since $$|P_3| = |P_4|,\ \ E_3^2 - M_3^2 = E_4^2 - M_4^2$$

Therefore $$dQ = \int \frac{\delta\left[E_3 + \left(E_3^2 - M_3^2 + M_4^2\right)^{1/2} - W\right] P_3 dE_3 d\Omega}{16\pi^2 E_4}$$

Now, $$\int dE_3 \delta\left(g(E_3)\right] = \left|\frac{dg}{dE_3}\right|^{-1}$$

$$g(E_3) = E_3 + \left(E_3^2 - M_3^2 + M_4^2\right)^{1/2} - W$$

$$\frac{dg}{d\text{E}_3} = 1 + \frac{\text{E}_3}{\text{E}_4} = \frac{W}{\text{E}_4}$$

Here $$\left(dQ\right)_{C.M.} = \frac{P_3 d\Omega}{16\pi^2 W} = \frac{P_f d\Omega}{16\pi^2 \sqrt{S}}$$

Now, consider dQ in the lab frame:

$$P_2 = (M_2, 0)$$

Integrating over d^3P_4, we have

$$dQ = \int \frac{\delta\left(\text{E}_3 + \text{E}_4 - \text{E}_1 - M_2\right) P_3 d\ \text{E}_3 d\Omega}{16\pi^2 \text{E}_4}$$

Now, $$\vec{P}_4 = \vec{P}_1 - \vec{P}_3 \qquad \left(\text{since} \quad \vec{P}_2 = 0\right)$$

$$\left|\vec{P}_4\right|^2 = \left|\vec{P}_1 - \vec{P}_3\right|$$

$$\text{E}_4^2 = M_4^2 + P_1^2 + P_3^2 - 2P_1P_3\cos\theta$$

For the δ-function, we have

$$g\left(\text{E}_3\right) = \text{E}_3 + \left(M_4^2 + P_1^2 + P_3^2 - 2P_1P_3\cos\theta\right)^{1/2} - \text{E}_1 - M_2$$

$$\frac{dg}{d\text{E}_3} = 1 + \left[2\text{E}_3 - 2\left(P_1\text{E}_3 / P_3\right)\cos\theta\right] / 2\text{E}_4$$

$$= \left[\text{E}_1 + M_2 - \left(P_1\text{E}_3 / P_3\right)\cos\theta\right] / \text{E}_4$$

Hence, $$\left(dQ\right)_{\text{lab}} = \frac{P_3 d\Omega}{16\pi^2\left[\text{E}_1 + M_2 - \left(P_1\text{E}_3 / P_3\right)\cos\theta\right]}$$

1.1 Mandelstam Variables

Before examining the $A + B \rightarrow C + D$ process in some detail, it is useful to consider the variables specifying such a reaction. There are various choices for variables, such as beam energy and scattering angle. However, it is advantageous to specify variables which are Lorentz invariant quantities. The Mandelstam variables (s, t and u) are defined as

$$\begin{aligned} s &= (P_A + P_B)^2 = (P_C + P_D)^2 && \text{total C.M. energy squared} \\ t &= (P_A - P_C)^2 = (P_D - P_B)^2 && \text{four momentum transfer squared} \\ u &= (P_A - P_D)^2 = (P_C - P_B)^2 && \end{aligned} \qquad (5.40)$$

Note that $P_A + P_B = P_C + P_D$.

s, t, u are not independent variables, since

$$s + t + u = M_A^2 + M_B^2 + M_C^2 + M_D^2 \qquad (5.41)$$

If $M_A = M_B = M_C = M_D = M$ ($e^-e^- \rightarrow e^-e^-$, $\pi^+\pi^+ \rightarrow \pi^+\pi^+$ for example), then in the C.M. frame we have

$$\begin{aligned} s &= 4(P^2 + M^2) \\ t &= -2P^2(1 - \cos\theta) \\ u &= -2P^2(1 + \cos\theta) \end{aligned} \qquad (5.42)$$

where P is the 3-momentum in the C.M. frame, and θ is the C.M. scattering angle. Note that $s > 0$, $t \leq 0$, $u \leq 0$.

The Mandelstam variables are very convenient in expressing one scattering process in terms of another related scattering process.

If one expresses the amplitude M for the process

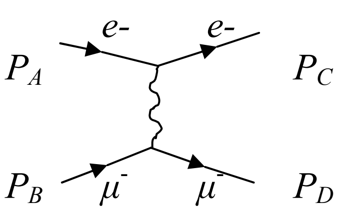
a) $P_A + P_B \rightarrow P_C + P_D$
as $M(s, t, u)$, then the other related processes have the amplitudes as follows:
b) $P_A + P_B \rightarrow P_D + P_C$: $M(s, u, t)$
c) $P_A + (-P_C) \rightarrow (-P_B) + P_D$: $M(t, s, u)$
d) $P_A + (-P_D) \rightarrow P_C + (-P_B)$: $M(u, t, s)$
e) $(-P_C) + (-P_D) \rightarrow (-P_A) + (-P_B)$: $M(s, t, u)$

As an example, take reaction a) as $e^-\mu^- \rightarrow e^-\mu^-$, then

a) $e^-\mu^- \rightarrow e^-\mu^-$: $M(s, t, u)$
b) $e^-\mu^- \rightarrow \mu^-e^-$: $M(s, u, t)$
c) $e^-e^+ \rightarrow \mu^+\mu^-$: $M(t, s, u)$
d) $e^-\mu^+ \rightarrow e^-\mu^+$: $M(u, t, s)$
e) $e^+\mu^+ \rightarrow e^+\mu^+$: $M(s, t, u)$

1.2 Spinless $e^-\mu^- \rightarrow e^-\mu^-$ Scattering

Now we consider the invariant amplitude M and the scattering cross-section for a 'spinless' $e^-\mu^- \rightarrow e^-\mu^-$ process.



Recall Equation 5.20

$$-iM = \left[ie\left(P_A + P_C\right)^\mu\right]\left(-i\frac{g_{\mu\nu}}{q^2}\right)\left[ie\left(P_B + P_D\right)^\nu\right]$$

first $e\gamma$ vertex — photon propagator — second $e\gamma$ vertex

$$M = -e\,\left(P_A + P_C\right) \bullet \left(P_B + P_D\right) \bullet \frac{1}{t} \qquad (5.43)$$

Now $\quad P_A + P_C = (P_A + P_B) + (P_C - P_B) = (P_A + P_B) + (P_A - P_D)$
$\quad P_B + P_D = (P_A + P_B) - (P_A - P_D)$

Therefore
$$M = -e^2\left[\left(P_A + P_B\right)^2 - \left(P_A - P_D\right)^2\right] \bullet \frac{1}{t} \qquad (5.44)$$
$$= -e^2\left(s-u\right)/t$$

Equations 5.39 and 5.44 give

$$\left.\frac{d\sigma}{d\Omega}\right)_{CM} = \frac{1}{64\pi^2 s}\;\frac{P_f}{P_i}\left(e^4\frac{\left(s-u\right)^2}{t^2}\right) \qquad (5.45)$$
$$= \frac{\alpha^2}{4s}\;\frac{P_f}{P_i}\;\frac{\left(s-u\right)^2}{t^2}$$

At high energies, masses are neglected, $P_i = P_f = P$ and

$$s = 4\left(P^2 + M^2\right) \simeq 4P^2$$
$$t = -2P^2\left(1-\cos\theta\right)$$
$$u = -2P^2\left(1+\cos\theta\right)$$

Hence,

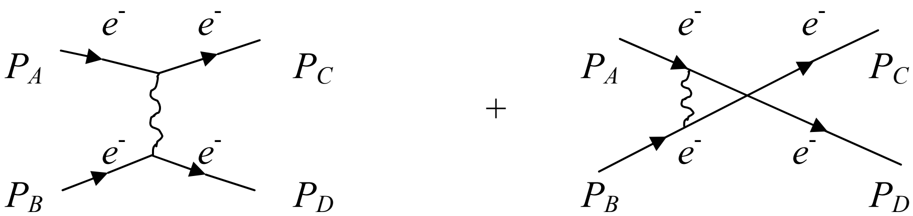
$$\frac{\left(s-u\right)^2}{t^2} = \frac{\left[4P^2 + 2P^2\left(1+\cos\theta\right)\right]^2}{\left[-2P^2\left(1-\cos\theta\right)\right]^2} = \frac{\left(3+\cos\theta\right)^2}{\left(1-\cos\theta\right)^2} \qquad (5.46)$$
$$= \frac{\left(2+2\cos^2\theta/2\right)^2}{\left(2\sin^2\theta/2\right)^2} = \frac{\left(1+\cos^2\theta/2\right)^2}{\sin^4\theta/2}$$

and

$$\left.\frac{d\sigma}{d\Omega}\right)_{CM} = \frac{\alpha^2}{4s}\left(\frac{1+\cos^2\theta/2}{\sin^2\theta/2}\right)^2$$

1.3 Spinless $e^-e^- \rightarrow e^-e^-$

Now consider spinless $e^-e^- \rightarrow e^-e^-$ scattering. There are two diagrams contributing to this process:



The first diagram is analogous to the diagram we considered earlier for $e^-\mu^- \to e^-\mu^-$. The second diagram reflects the fact that one can not tell if P_C originates from P_A or from P_B.

The invariant amplitude is a sum of these two diagrams

$$-iM_{e^-e^-} = -i\left[-e^2\frac{(P_A+P_C)\bullet(P_B+P_D)}{(P_D-P_B)^2} - e^2\frac{(P_A+P_D)\bullet(P_B+P_C)}{(P_C-P_B)^2}\right] \quad (5.48)$$

Note that $M_{e\text{-}e\text{-}}$ is now symmetric with respect to the exchange of $P_C \leftrightarrow P_D$, as well as the exchange of $P_A \leftrightarrow P_B$. This is a consequence that we assume e^- is spinless and following Bose statistics. Otherwise, the amplitude should be antisymmetric with respect to these exchanges.

In terms of the Mandelstam variables, Equation 5.48 can be expressed as

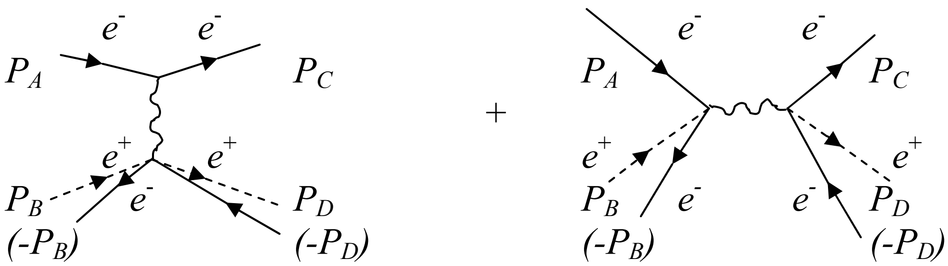
$$M_{e^-e^-} = -e^2\left(\frac{s-u}{t}\right) - e^2\left(\frac{s-t}{u}\right) \quad (5.49)$$

The second term in Equation 5.49 is obtained by $t \leftrightarrow u$ exchange in the first term, as one expects.

$$\left(\frac{d\sigma}{d\Omega}\right)_{CM} = \frac{\alpha^2}{s}\,\frac{(4-\sin^2\theta)^2}{\sin^4\theta}$$

1.4 Spinless $e^-e^+ \to e^-e^+$

There are two diagrams contributing to this reaction:



The first diagram is the exchange diagram analogous to $e^-\mu^- \rightarrow e^-\mu^-$. The second diagram is an annihilation diagram. Note that the e^+ lines are replaced by e^- lines with opposite momenta. The corresponding invariant amplitudes are

$$-iM_{e^-e^+} = -i\left[-e^2 \frac{(P_A + P_C) \bullet (-P_D - P_B)}{(P_D - P_B)^2} - e^2 \frac{(P_A - P_B) \bullet (-P_D + P_C)}{(P_C - P_D)^2} \right] \qquad (5.50)$$

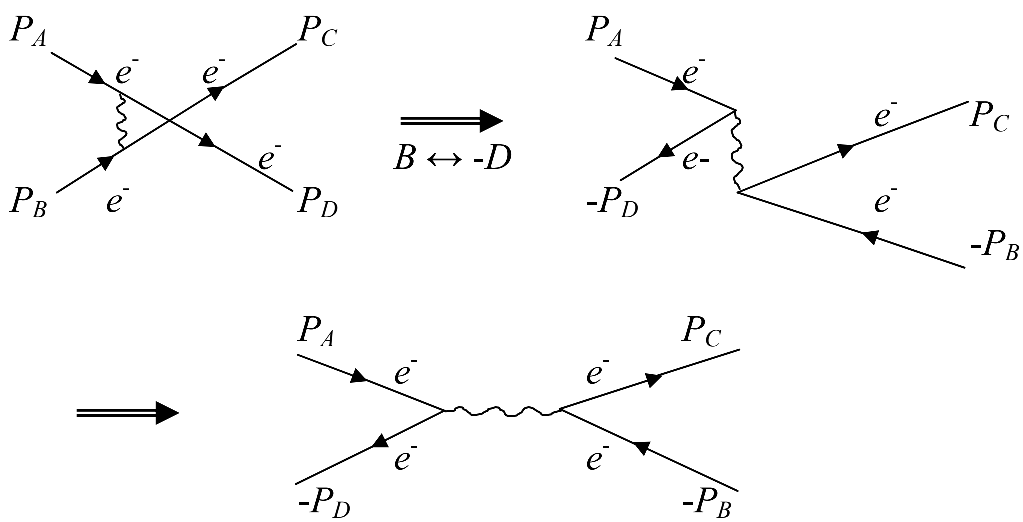
At each vertex $P + P'$ corresponds to the incoming e^- momentum P and the outgoing e^- momentum P'.

In terms of the Mandelstam variables, Equation 5.50 can be written as

$$\begin{aligned} M_{e^-e^+} &= -e^2\left(\frac{u-s}{t}\right) - e^2\left(\frac{u-t}{s}\right) \\ &= e^2\left(\frac{s-u}{t}\right) + e^2\left(\frac{t-u}{s}\right) \end{aligned} \qquad (5.51)$$

Equation 5.51 can be obtained from Equation 5.49 ($e^-e^- \rightarrow e^-e^-$) by interchanging ($s \leftrightarrow u$).

Although the annihilation diagram for $e^-e^+ \rightarrow e^-e^+$ has a different appearance compared with the second exchange diagram in the $e^-e^- \rightarrow e^-e^-$ reaction, these two diagrams are actually related by the $B \leftrightarrow -D$ interchange. This can be seen graphically by interchanging $B \leftrightarrow - D$ for the $e^-e^- \rightarrow e^-e^-$ diagram.



which ends up as the annihilation diagram for the $e^- e^+ \rightarrow e^- e^+$.

It is interesting to note that the cross-section for $e^- e^- \rightarrow e^- e^-$ scattering (Equation 5.49) diverges at $\theta = 0^o$ and $\theta = 180^o$, corresponding to $t = 0$ and $u = 0$. Since t and u are the invariant masses of the exchanged virtual photons for the two diagrams of the $e^- e^- \rightarrow e^- e^-$ scattering, a vanishing mass of the virtual photon implies that the range of the interaction becomes infinite. Hence the cross-section diverges.

For the annihilation diagram, the corresponding amplitude does not diverge, since the virtual photon has an invariant mass greater than $2M_e$ and cannot be zero.

2. EM Interactions of spin-½ particles

We follow a similar procedure as the spin-0 case to obtain the expression for the invariant amplitude.

For a spin-½ charged particle interacting with an EM field, the Dirac equation

$$\left(\gamma_\mu P^\mu - m\right)\psi = 0 \tag{5.52}$$

becomes (after the $P^\mu \rightarrow P^\mu + eA^\mu$ substitution)

$$\left(\gamma_\mu P^\mu - m\right)\psi = \rightarrow \left(-e\gamma_\mu A^\mu\right)\psi = \left(\gamma^0 V\right)\psi \tag{5.53}$$

where

$$\gamma^0 V = -e\gamma_\mu A^\mu \tag{5.54}$$

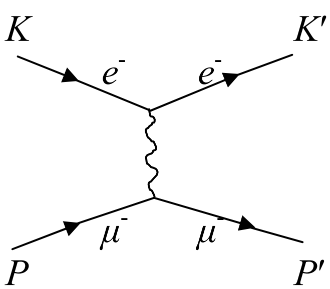
The transition amplitude T_{fi} is given as

$$\begin{aligned} T_{fi} &= -i\int \psi_f^+(x) V(x) \psi_i(x) d^4x \\ &= ie\int \overline{\psi}_f(x)\gamma_\mu A^\mu \psi_i(x) d^4x \\ &= -i\int j_\mu A^\mu d^4x \end{aligned} \tag{5.55}$$

where the current density j_μ for the $i \to f$ transition is

$$j_\mu = -e\overline{\psi}_f \gamma_\mu \psi_i \tag{5.56}$$

Now, consider the $e^-\mu^- \to e^-\mu^-$ scattering (with spin-½ e^- and μ^-)



Following similar steps as for the Klein-Gordon equation, one can deduce

$$-iM = \left(ie\overline{u}(K')\gamma^\mu u(K)\right)\left(\frac{ig_{\mu\nu}}{q^2}\right)\left(ie\overline{u}(P')\gamma_\nu u(P)\right)$$

first $e\gamma$ vertex — γ-propagator — second $\mu\gamma$ vertex

2.1 <u>$e^-\mu^- \to e^-\mu^-$ Scattering</u>

$$M = -e^2\overline{u}(K')\gamma^\mu u(K)\frac{1}{q^2}\overline{u}(P')\gamma_\mu u(P) \tag{5.58}$$

For measurements using unpolarized e^- and μ^-, the scattering cross-section should be an incoherent sum over the various spin states of e^-, μ^-, and averaged over the initial e^-, μ^- spins:

$$\overline{|M|^2} = \frac{1}{(2S_A+1)(2S_B+1)}\sum_{\substack{\text{spin}\\\text{states}}}|M|^2 = \frac{1}{4}\sum_{\substack{\text{spin}\\\text{states}}}|M|^2 \tag{5.59}$$

Equations 5.58 and 5.59 give

$$\overline{|M|^2} = \frac{1}{4}\sum_{spin}\frac{e^4}{q^4}\left(\bar{u}(K')\gamma^\mu u(K)\bar{u}(P')\gamma_\mu u(P)\right)\left(\bar{u}(K')\gamma^\nu u(K)\bar{u}(P')\gamma_\nu u(P)\right)^* \quad (5.60)$$

$\overline{|M|^2}$ can be viewed as a contraction of two lepton tensors

$$\overline{|M|^2} = \frac{e^4}{q^4} L_e^{\mu\nu} L_{\mu\nu}^{muon} \quad (5.61)$$

For the electron tensor, $L_e^{\mu\nu}$, we have

$$L_e^{\mu\nu} = \frac{1}{2}\sum_{s,s'}\left(\bar{u}^{s'}(K')\gamma^\mu u^s(K)\right)\left(\bar{u}^{s'}(K')\gamma^\nu u^s(K)\right)^* \quad (5.62)$$

Since $\bar{u}(K')\gamma^\nu u(K)$ is a number, its complex conjugate is identical to its Hermitian conjugate. Therefore

$$\left[\bar{u}^{s'}(K')\gamma^\nu u^s(K)\right]^+ = u^s(K)^+\left(\gamma^\nu\right)^+\gamma^0 u^{s'}(K') = \bar{u}^s(K)\gamma^\nu u^{s'}(K') \quad (5.63)$$

where we have used $\left(\gamma^\nu\right)^+\gamma^0 = \gamma^0\gamma^\nu$ relation.

Equation 5.63 shows that the operation of complex conjugate on $\bar{u}(K')\gamma^\nu u(K)$ is simply equivalent to interchanging (K, S) and (K′, S′).

From Equations 5.62 and 5.63, we obtain

$$L_e^{\mu\nu} = \frac{1}{2}\sum_{s,s'}\left(\bar{u}^{s'}(K')\gamma^\mu u^s(K)\right)\left(\bar{u}^s(K)\gamma^\nu u^{s'}(K')\right) \quad (5.64)$$

Each term on the right-hand side of Equation 5.64 is a product of two numbers. It is useful to view Equation 5.64 in a somewhat different fashion:

$$\bar{u}^{s'}(K')\gamma^{\mu}u^{s}(K)\bar{u}^{s}(K)\gamma^{\nu}u^{s'}(K') = \left[\bar{u}^{s'}(K')\gamma^{\mu}u^{s}(K)\bar{u}^{s}(K)\gamma^{\nu}\right]\left[u^{s'}(K')\right] \quad (5.65)$$

Now, the right-hand side of Equation 5.65 corresponds to a product of a column 4 x 1 matrix by a row 1 x 4 matrix:

$$\begin{pmatrix} a_1 & a_2 & a_3 & a_4 \end{pmatrix}\begin{pmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{pmatrix} = A \bullet B \quad (5.66)$$

One can invert the order of *A* and *B*, and Equation 5.66 becomes

$$\begin{pmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{pmatrix} \bullet \begin{pmatrix} a_1 & a_2 & a_3 & a_4 \end{pmatrix} = \begin{pmatrix} a_1b_1 & a_2b_1 & a_3b_1 & a_4b_1 \\ a_1b_2 & a_2b_2 & a_3b_2 & a_4b_2 \\ a_1b_3 & a_2b_3 & a_3b_3 & a_4b_3 \\ a_1b_4 & a_2b_4 & a_3b_4 & a_4b_4 \end{pmatrix} \quad (5.67)$$

Therefore

$$A \bullet B = \begin{pmatrix} a_1 & a_2 & a_3 & a_4 \end{pmatrix} \bullet \begin{pmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{pmatrix} = T_r \begin{pmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{pmatrix} \bullet \begin{pmatrix} a_1 & a_2 & a_3 & a_4 \end{pmatrix} \quad (5.68)$$

Using Equation 5.68, Equation 5.65 becomes (after moving $u^{s'}(K')$ to the front)

$$\bar{u}^{s'}(K')\gamma^{\mu}u^{(s)}(K)\bar{u}^{s}(K)\gamma^{\nu}u^{s'}(K') = \left[u^{s'}(K')\bar{u}^{s'}(K')\gamma^{\mu}u^{s}(K)\bar{u}^{s}(K)\gamma^{\nu}\right] \quad (5.69)$$

Now we can use the completeness relation for the Dirac spinor

$$\sum_{s'} u^{s'}(K')\bar{u}^{s'}(K') = \not{K}' + M$$

to evaluate Equation 5.64 and we obtain

$$\begin{aligned} L_e^{\mu\nu} &= \frac{1}{2}\sum_{s,s'} T_r\left(u^{s'}(K')\bar{u}^{s'}(K')\gamma^\mu u^s(K)\bar{u}^s(K)\gamma^\nu\right) \\ &= \frac{1}{2}T_r\left[\left(\not{K}' + M\right)\gamma^\mu\left(\not{K} + M\right)\gamma^\nu\right] \end{aligned} \tag{5.70}$$

The evaluation of the $L_e^{\mu\nu}$ is now reduced to an evaluation of traces of products of γ matrices. Several useful trace theorems as well as contraction theorems can be readily derived.

Trace Theorems:

$$\begin{aligned} &T_r\left(\text{odd number of } \gamma^\mu\right) = 0 \\ &T_r\left(\gamma^\mu\gamma^\nu\right) = 4g^{\mu\nu} \\ &T_r\left(\gamma^\mu\gamma^\nu\gamma^\lambda\gamma^\sigma\right) = 4\left(g^{\mu\nu}g^{\lambda\sigma} - g^{\mu\lambda}g^{\nu\sigma} + g^{\mu\sigma}g^{\nu\lambda}\right) \end{aligned} \tag{5.71}$$

Contraction Theorems:

$$\begin{aligned} &\gamma_\mu\gamma^\mu = 4 \\ &\gamma_\mu\gamma^\nu\gamma^\mu = -2\gamma^\nu \\ &\gamma_\mu\gamma^\nu\gamma^\lambda\gamma^\mu = 4g^{\nu\lambda} \\ &\gamma_\mu\gamma^\nu\gamma^\lambda\gamma^\sigma\gamma^\mu = -2\gamma^\sigma\gamma^\lambda\gamma^\nu \end{aligned} \tag{5.72}$$

From Equation 5.71, the electron tensor becomes

$$\begin{aligned} L_e^{\mu\nu} &= \frac{1}{2}T_r\left(\not{K}'\gamma^\mu\not{K}\gamma^\nu\right) + \frac{1}{2}M^2 T_r\left(\gamma^\mu\gamma^\nu\right) \\ &= 2\left[K'^\mu K^\nu + K'^\nu K^\mu - \left(K \bullet K'\right)g^{\mu\nu} + M^2 g^{\mu\nu}\right] \end{aligned} \tag{5.73}$$

Similarly, the muon tensor becomes

$$L_e^{\mu\nu} = 2\left[P'_\mu P_\nu + P'_\nu P_\mu - \left(P \bullet P'\right)g_{\mu\nu} + M^2 g_{\mu\nu}\right] \tag{5.74}$$

and finally

$$\overline{|M|^2} = \frac{8e^4}{q^4}\Big[(K'\bullet P')(K\bullet P) + (K'\bullet P)(K\bullet P') - M^2(P\bullet P') \\ - M^2(K\bullet K') + 2M^2M^2\Big] \tag{5.75}$$

A useful relation for carrying out Lepton tensor contraction is

$$q^\mu L_{\mu\nu} = q^\nu L_{\mu\nu} = 0 \tag{5.76}$$

Equation 5.76 follows form current conservation since

$$\partial^\mu j_\mu = 0$$
$$\partial^\mu\left(\overline{u}(K')\gamma_\mu u(K)e^{-i(K-K')\bullet x}\right) = 0$$

Therefore

$$q^\mu\left(\overline{u}(K')\gamma_\mu u(K)\right) = 0$$

and

$$q^\mu L_{\mu\nu} = 0$$

Equation 5.76 can also be proven by noting

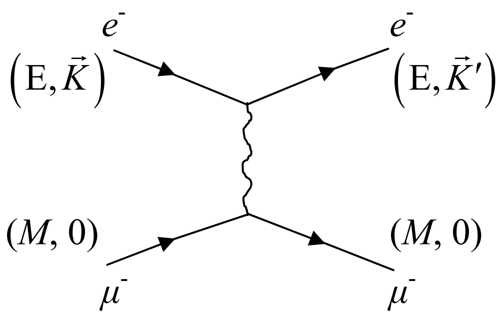
$$q^\mu\left(\overline{u}(K')\gamma_\mu u(K)\right) = \overline{u}(K')\not{q}u(K)$$
$$= \overline{u}(K')\left(\not{K} - \not{K}'\right)u(K) = 0$$
$$\text{(since } \left(\not{K} - m\right)u(K) = 0;\ \ \overline{u}(K')\left(\not{K}' - m\right) = 0\text{)}$$

Hence

$$\overline{u}(K')\left(\not{K} - \not{K}'\right)u(K) = \overline{u}(K')mu(K) - \overline{u}(K')mu(K) = 0$$

We consider three limiting cases for $e^-\mu^- \to e^-\mu^-$ scattering:

<u>a) *M* >> *m*</u>



For a very massive 'μ^-', there is no recoil, and the μ four-vector in the final state remains $(M, 0)$.

Also, $|\vec{K}'| = |\vec{K}|$

Note that in this case, C.M. frame is the same as lab frame.

Recall Equation 5.75

$$\overline{|M|^2} = \frac{8e^4}{q^4}\Big[(K'\bullet P')(K\bullet P) + (K'\bullet P)(K\bullet P') - M^2(P\bullet P') - M^2(K\bullet K') + 2M^2M^2\Big] \tag{5.75}$$

To evaluate Equation 5.75, we note

$$\begin{aligned} &K'\bullet P' = K\bullet P = M\mathrm{E} \\ &K'\bullet P = K\bullet P' = M\mathrm{E} \\ &P\bullet P' = M^2 \\ &K\bullet K' = \mathrm{E}^2 - \vec{K}\bullet\vec{K}' = |K|^2 + M^2 - |K|^2\cos\theta = M^2 + 2|K|^2\sin^2\frac{\theta}{2} \\ &q^2 = (K-K')^2 = -(\vec{K}-\vec{K}')^2 = -2|K|^2(1-\cos\theta) = -4|K|^2\sin^2\frac{\theta}{2} \end{aligned} \tag{5.77}$$

Equation 5.75 becomes

$$\overline{|M|^2} = \frac{8e^4}{16|K|^4 \sin^4\frac{\theta}{2}}\Big[M^2E^2 + M^2E^2 - m^2M^2 - M^2m^2$$

$$\qquad\qquad - 2M^2|K|^2\sin^2\frac{\theta}{2} + 2m^2M^2\Big] \tag{5.78}$$

$$= \frac{e^4}{|K|^4\sin^4\frac{\theta}{2}}\left[M^2E^2 - M^2|K|^2\sin^2\frac{\theta}{2}\right]$$

In the C.M. frame, the differential cross-section can be written as

$$\frac{d\sigma}{d\Omega} = \frac{1}{64\pi^2 s}\overline{|m|^2}\frac{|K'|}{|K|} = \frac{\alpha^2E^2}{4|K|^4\sin^4\theta/2}\left(1 - v^2\sin^2\frac{\theta}{2}\right) \tag{5.79}$$

where we have used the following relations

$$s = (E + M)^2 - K^2 = E^2 - K^2 + M^2 + 2ME = m^2 + M^2 + 2ME \simeq M^2$$
$$|K| = vE$$
$$|K'| = |K|$$
$$\alpha = e^2/4\pi$$

Equation 5.79 is the Mott scattering formula, representing the scattering of a spin-½ charged particle off a static field.

Note that at the relativistic limit, $v \rightarrow 1$ and Equation 5.79 becomes

$$\frac{d\sigma}{d\Omega} = \frac{\alpha^2E^2}{4|K|^4\sin^4\theta/2}\cos^2\frac{\theta}{2} \tag{5.80}$$

In this case, electron is forbidden to scatter to 180° due to helicity conservation.

b) Muon can recoil, but E >> m and set $m = 0$

Equation 5.75 becomes

$$\overline{|M|^2} = \frac{8e^4}{q^4}\left[(K'\bullet P')(K\bullet P) + (K'\bullet P)(K\bullet P') - M^2(K\bullet K')\right]$$
$$= \frac{8e^4}{q^4}\left[(K\bullet K')(K\bullet P - K'\bullet P) + 2(K'\bullet P)(K\bullet P) - M^2(K\bullet K')\right]$$

where we have expressed P' as $P' = K + P - K'$, and used $K^2 = K'^2 = 0$.

Using

$$K\bullet K' = -q^2/2$$
$$P = (M,0)$$
$$q^2 = -4\mathrm{E}\mathrm{E}'\sin^2\theta/2$$
$$\nu = \mathrm{E} - \mathrm{E}' = -q^2/2M$$

we obtain

$$\overline{|M|^2} = \frac{8e^4}{q^4}\left[-q^2/2\,(\mathrm{E}M - \mathrm{E}'M) + 2(\mathrm{E}M)(\mathrm{E}'M) + 1/2\,M^2q^2\right]$$
$$= \frac{8e^4}{q^4}\left[2M^2\mathrm{E}\mathrm{E}'\right]\left[\cos^2\frac{\theta}{2} - \frac{q^2}{2M^2}\sin^2\frac{\theta}{2}\right] \tag{5.81}$$

Finally, we obtain

$$\frac{d\sigma}{d\Omega} = \frac{\alpha^2}{4\mathrm{E}^2\sin^4\theta/2}\left(\frac{\mathrm{E}'}{\mathrm{E}}\right)\left[\cos^2\frac{\theta}{2} - \frac{q^2}{2M^2}\sin^2\frac{\theta}{2}\right] \tag{5.82}$$

(see pp. 131-132 of Halzen & Martin for the derivation of Equation 5.82 from Equation 5.81)

Note that the $\sin^2\frac{\theta}{2}$ term in Equation 5.82 allows the incident electron to scatter to 180o. This term is due to the magnetic moment of the muon, allowing spin-flip of the incident electron. This can be further illustrated by noting that for $e^-\pi^- \rightarrow e^-\pi^-$ scattering

$$\frac{d\sigma}{d\Omega} = \frac{\alpha^2}{4\mathrm{E}^2 \sin^4 \theta/2}\left(\frac{\mathrm{E}'}{\mathrm{E}}\right)\cos^2\frac{\theta}{2} \tag{5.83}$$

<u>Note added:</u>

One can derive Equation 5.82 as follows:

Equations 5.33, 5.35, give

$$\left(\frac{d\sigma}{d\Omega}\right)_{lab} = \frac{\overline{|M|^2}}{4P_1 m_2}\ \frac{P_3}{16\pi^2\left[\mathrm{E}_1 + M_2 - \left(P_1\mathrm{E}_3/P_3\right)\cos\theta\right]}$$

$$\mathrm{E}_1 + M_2 - \left(P_1\mathrm{E}_3/P_3\right)\cos\theta = \mathrm{E}_1 + M_2 - \mathrm{E}_1\cos\theta$$

$$(\text{since } P_1 = \mathrm{E}_1,\ P_3 = \mathrm{E}_3 \text{ when } m_1 \rightarrow 0)$$

$$= \mathrm{E} + M - \mathrm{E}\cos\theta = 2\mathrm{E}\sin^2\frac{\theta}{2} + M$$

but

$$\nu = \mathrm{E} - \mathrm{E}' = -q^2/2M$$

and

$$q^2 = -4\mathrm{E}\mathrm{E}'\sin^2\theta/2$$

Therefore

$$2\mathrm{E}\sin^2\frac{\theta}{2} = \frac{-q^2}{2\mathrm{E}'} = \frac{2M\left(\mathrm{E}-\mathrm{E}'\right)}{2\mathrm{E}'} = M\left(\frac{\mathrm{E}}{\mathrm{E}'} - 1\right)$$

$$\left(\frac{d\sigma}{d\Omega}\right)_{lab} = \frac{\overline{|M|^2}}{4P_1 m_2}\ \frac{P_3}{16\pi^2\left(\mathrm{E}/\mathrm{E}'\,M\right)}$$

Using Equation 5.81, we obtain

$$\left(\frac{d\sigma}{d\Omega}\right)_{lab} = \frac{\alpha^2}{4E^2\sin^4\theta/2}\left(\frac{E'}{E}\right)\left[\cos^2\frac{\theta}{2} - \frac{q^2}{2M^2}\sin^2\frac{\theta}{2}\right]$$

c) Relativistic Limit (neglecting both m^2 and M^2)

In this limit, Equation 5.75 simplifies to

$$\overline{|M|^2} = \frac{8e^4}{q^4}\left[\left(K'\bullet P'\right)\left(K\bullet P\right)+\left(K'\bullet P\right)\left(K\bullet P'\right)\right] \qquad (5.81)$$

Neglecting m^2 and M^2, the Mandelstam variables become

$$\begin{aligned} s &= \left(K+P\right)^2 \simeq 2K\bullet P = 2K'\bullet P' \\ t &= \left(K-K'\right)^2 \simeq -2K\bullet K' = -2P\bullet P' \\ u &= \left(K-P'\right)^2 \simeq -2K\bullet P' = -2K'\bullet P \end{aligned}$$

Equation 5.81 becomes

$$\overline{|M|^2} = 2e^4\frac{s^2+u^2}{t^2} \qquad (5.82)$$

and the C.M. cross-section is

$$\left(\frac{d\sigma}{d\Omega}\right)_{CM} = \frac{1}{64\pi^2 s}\frac{P_f}{P_i}\overline{|M|^2} = \frac{\alpha^2}{2s}\left(\frac{1+\cos^4 \theta/2}{\sin^4 \theta/2}\right) \qquad (5.83)$$

where we use the following expressions in the C.M. frame:

$s = 4K^2$ $\qquad$ $t = -2K^2\,(1-\cos\theta)$ $\qquad$ $u = -2K^2\,(1+\cos\theta)$

Although one cannot check the expression for $e^-\mu^- \rightarrow e^-\mu^-$ scattering at the high energy limit, one can consider several related reactions which can be and have been studied experimentally.

c.1) $e^-e^\pm \to \mu^-\mu^\pm$

The scattering amplitude for this process can be obtained from $e^-\mu^- \to e^-\mu^-$ by interchanging $s \leftrightarrow t$ first, giving $e^-e^+ \to \mu^-\mu^+$, followed by $t \leftrightarrow u$ exchange, resulting in $e^-e^+ \to \mu^-\mu^+$.

Hence from Equation 5.82, one obtains for $e^-e^+ \to \mu^-\mu^+$

$$\overline{|M|^2} = 2e^4 \frac{u^2+t^2}{s^2} \tag{5.84}$$

The differential cross-section for $e^-e^+ \to \mu^-\mu^+$ is

$$\left(\frac{d\sigma}{d\Omega}\right)_{e^-e^+\to\mu^-\mu^+} = \frac{\alpha^2}{2s} \frac{4K^4\left(2+2\cos^2\theta\right)}{16K^4} = \frac{\alpha^2}{4s}\left(1+\cos^2\theta\right) \tag{5.85}$$

and the total cross-section is

$$\sigma\left(e^-e^+ \to \mu^-\mu^+\right) = \frac{4\pi\alpha^2}{3s} \tag{5.86}$$

Both the $(1 + \cos^2\theta)$ angular distribution in Equation 5.85 and the $\frac{1}{s}$ dependence of the total cross-section in Equation 5.86 are well confirmed by experiments.

c.2) $e^-e^+ \to q\bar{q}$

This process is analogous to the $e^-e^+ \to \mu^-\mu^+$ scattering. An important difference, apart from the factor Q_q^2 for the quark charge, is the color factor of 3 to account for the 3 colors for the quarks.

$$\begin{aligned} \sigma\left(e^-e^+ \to q\bar{q}\right) &= \frac{4\pi\alpha^2}{3s} \times Q_q^2 \times 3 \\ &= 3 \times Q_q^2\ \sigma\left(e^-e^2 \to \mu^-\mu^+\right) \end{aligned} \tag{5.87}$$

Experimentally, quark-antiquark pairs are not observed. Instead, the hadrons into which the $q\overline{q}$ hadronize are detected. One can measure the R factor, defined as

$$R = \frac{\sigma\left(e^- e^+ \rightarrow \text{hadrons}\right)}{\sigma\left(e^- e^+ \rightarrow \mu^- \mu^+\right)} = 3\sum_q Q_q^2 \tag{5.88}$$

Depending on the C.M. energy, various $q\overline{q}$ channels may be open. One expects R to be:

$$R = 3\left[\left(\frac{2}{3}\right)^2 + \left(\frac{1}{3}\right)^2 + \left(\frac{1}{3}\right)^2\right] = 2$$

if u, d, s quark pairs can be produced.

If the C.M. is above the threshold for charm-quark pair production, then

$$R = 2 + 3\left(\frac{2}{3}\right)^2 = \frac{10}{3}$$

and (5.89)

$$R = \frac{10}{3} + 3\left(\frac{1}{3}\right)^2 = \frac{11}{3}$$

once the $b\overline{b}$ threshold is passed.

The experimental data are in good agreement with the expectations from Equation 5.89.

Note that if there is no color factor of 3, the predicted R would be in a strong disagreement with the data.

c.3) $q\overline{q} \rightarrow e^- e^+$, $q\overline{q} \rightarrow \mu^- \mu^+$

This is the inverse reaction of $e^- e^+ \rightarrow q\overline{q}$. It can be studied experimentally in hadron-hadron interaction, in which a quark from one hadron interacts with the antiquark in the other hadron. This process is also called the Drell-Yan process.

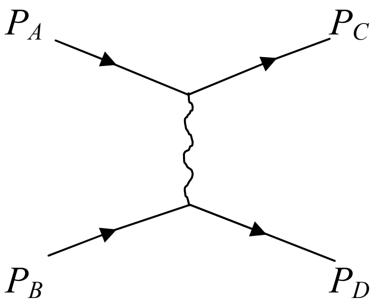
The cross-section for this process is analogous to the $e^- e^+ \rightarrow q\overline{q}$, with an important difference. The color degree of freedom for the quarks / antiquarks implies that only $q - \overline{q}$ with matched color (blue-antiblue, for example) can annihilate. Hence the cross-section is

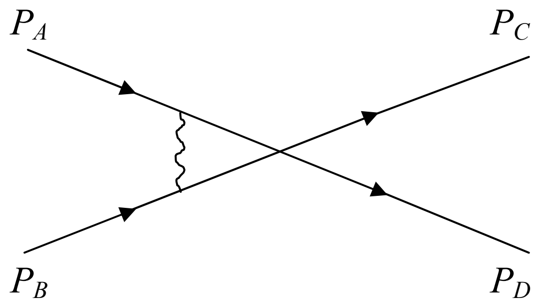
$$\sigma\left(q\overline{q} \rightarrow e^- e^+\right) = \sigma\left(e^- e^+ \rightarrow \mu^- \mu^+\right)\frac{Q_q^2}{3} \qquad (5.90)$$

Also, the angular distributions show a 1 + $\cos^2\theta$ dependence. Both the cross-section and the 1 + $\cos^2\theta$ dependence have been verified experimentally.

2.2) $e^-e^- \rightarrow e^-e^-$ Scattering (Møller Scattering)

There are now two diagrams contributing:





The scattering amplitude combines both contributions with a negative sign between them.

$$M = -e^2 \frac{(\bar{u}_C\gamma^\mu u_A)(\bar{u}_D\gamma_\mu u_B)}{(P_A - P_C)^2} + e^2 \frac{(\bar{u}_D\gamma^\mu u_A)(\bar{u}_C\gamma_\mu u_B)}{(P_A - P_D)^2} \tag{5.91}$$

M is antisymmetric with respect to interchange of identical fermions ($A\leftrightarrow B$, $C\leftrightarrow D$). This reflects Fermi statistics. For identical bosons, *M* is symmetric with respect to interchange of identical bosons.

The contribution to $\overline{|M|^2}$ from the first diagram is identical to that of $e^-\mu^- \rightarrow e^-\mu^-$ (Equation 5.82)

$$\overline{|M|^2}_{\text{direct}} = 2e^4 \frac{s^2 + u^2}{t^2} \tag{5.92}$$

The contribution from the second diagram is obtained from Equation 5.92 by t↔ u interchange:

$$\overline{|M|^2}_{\text{exchange}} = 2e^4 \frac{s^2 + t^2}{u^2} \tag{5.93}$$

The evaluation of the interference terms is as follows:

$$\overline{|M|^2}_{\text{interf.}} = \frac{1}{(2s_A + 1)(2s_B + 1)} \left(-e^4\right) \frac{(\text{Int. 1}) + (\text{Int. 2})}{tu} \tag{5.94}$$

First interference term (Int. 1) is

$$\text{Int. } 1 = \sum_{\substack{s,s' \\ \lambda,\lambda'}} \left(\bar{u}_C^{s'}(K')\gamma^\mu u_A^s(K)\right)\left(\bar{u}_D^{\lambda'}(P')\gamma_\mu u_B^\lambda(P)\right) \left(\bar{u}_A^s(K)\gamma^\nu u_D^{\lambda'}(P')\right)\left(\bar{u}_B^\lambda(P)\gamma_\nu u_C^{s'}(K')\right) \tag{5.95}$$

One can interchange the second and third term of Equation 5.95, since both terms are just numbers. Also, moving $u_C^{s'}(K')$ term to the front, one can evaluate the trace:

$$\begin{aligned} \text{Int. } 1 &= \sum_{\substack{s,s' \\ \lambda,\lambda'}} Tr\Big[u_C^{s'}(K')u_C^{s'}(K')\gamma^\mu u_A^s(K)\bar{u}_A^s(K)\gamma^\nu u_D^{\lambda'}(P') \\ &\qquad \bar{u}_D^{\lambda'}(P')\gamma_\mu u_B^\lambda(P)\bar{u}_B^\lambda(P)\gamma_\nu\Big] \\ &= Tr\Big[\left(\not{K}' + m\right)\gamma^\mu\left(\not{K} + m\right)\gamma^\nu\left(\not{P}' + m\right)\gamma_\mu\left(\not{P} + m\right)\gamma_\nu\Big] \\ &= Tr\Big[\not{K}'\gamma^\mu \not{K}\gamma^\nu \not{P}'\gamma_\mu \not{P}\gamma_\nu\Big] \quad (\text{neglecting } m) \end{aligned} \tag{5.96}$$

using contraction relations

$$\gamma_\mu \not{a}\not{b}\not{c}\gamma^\mu = -2\not{c}\not{b}\not{a}$$

and

$$\gamma_\mu \not{a}\not{b}\gamma^\mu = 4a\bullet b$$

and the trace theorem

$$Tr\not{a}\not{b} = 4a\bullet b$$

Equation 5.96 becomes

$$\begin{aligned} \text{Int. } 1 &= Tr\Big[\not{K}'\gamma^\mu \not{K}(-2)\not{P}\gamma_\mu \not{P}'\Big] \\ &= Tr\Big[\not{K}'(-2)(4)(K\bullet P)\not{P}'\Big] \\ &= -32(K\bullet P)(K'\bullet P') \\ &= -8s^2 \end{aligned} \tag{5.97}$$

The second interference term (Int. 2) is identical to Int. 1, since (Int. 2)* = Int. 1 is a real quantity.

Collecting Equation 5.94 and Equation 5.97, we obtain

$$\overline{|M|^2_{\text{int}}} = 4e^4 \frac{s^2}{tu} \tag{5.98}$$

From the $e^- e^- \to e^- e^-$ Møller scattering formula, one can readily obtain the $e^- e^+ \to e^- e^+$ Bhabha scattering formula by crossing ($s \leftrightarrow u$). The scattering amplitudes for $e^- e^- \to e^- e^-$, $e^- e^+ \to e^- e^+$, and $e^- \mu^- \to e^- \mu^-$ and $e^- e^+ \to \mu^+ \mu^-$ can be summarized in the following table:

$$\overline{|M|^2} \big/ 2e^4$$

	Forward	Interference	Backward
$e^- e^- \to e^- e^-$ (Møller)	$\frac{s^2+u^2}{t^2}$	$\frac{2s^2}{tu}$	$\frac{s^2+t^2}{u^2}$
($s \leftrightarrow u$)		$u \leftrightarrow t$	
$e^- e^+ \to e^- e^+$	$\frac{s^2+u^2}{t^2}$	$\frac{2u^2}{ts}$	$\frac{u^2+t^2}{s^2}$
$e^- \mu^- \to e^- \mu^-$	$\frac{s^2+u^2}{t^2}$		
($s \leftrightarrow t$)			
$e^- e^+ \to \mu^+ \mu^-$			$\frac{u^2+t^2}{s^2}$

2.3) Helicity Conservation

As discussed earlier, at high energy (E >> m), the following relation holds for the Dirac spinors:

$$\gamma^5 u(p) \simeq \vec{\Sigma} \bullet \hat{p}\; u(p) = \begin{pmatrix} \vec{\sigma} \bullet \hat{p} & 0 \\ 0 & \vec{\sigma} \bullet \hat{p} \end{pmatrix} u(p) \tag{5.99}$$

$\vec{\Sigma} \bullet \hat{p}$ is the helicity operator. Equation 5.99 implies

$$\gamma^5 u(p) = (+1)u(p) \text{ if } u(p) \text{ has helicity} = +1$$
$$\gamma^5 u(p) = (-1)u(p) \text{ if } u(p) \text{ has helicity} = -1 \qquad (5.100)$$

The $\frac{1}{2}(1 \pm \gamma^5)$ become the helicity projection operators. For example, from Equation 5.100, it follows that

$$\frac{1}{2}(1 \pm \gamma^5)u(p) = 0 \text{ if } u(p) \text{ has helicity} = +1$$
$$= u(p) \text{ if } u(p) \text{ has helicity} = -1 \qquad (5.101)$$

Similarly (note that $\frac{1}{2}(1-\gamma^5)\left[\frac{1}{2}(1-\gamma^5)u(p)\right] = \left[\frac{1}{2}(1-\gamma^5)u(p)\right]$. Hence $\left[\frac{1}{2}(1-\gamma^5)u(p)\right]$ has helicity = -1)

$$\frac{1}{2}(1 + \gamma^5)u(p) = 0 \text{ if } u(p) \text{ has helicity} = -1$$
$$= u(p) \text{ if } u(p) \text{ has helicity} = +1 \qquad (5.102)$$

Since $\frac{1}{2}(1-\gamma^5)$ and $\frac{1}{2}(1+\gamma^5)$ project out the left- and right-handedness of *u*(*p*), we write

$$U_R = \frac{1}{2}(1+\gamma^5)u \qquad U_L = \frac{1}{2}(1-\gamma^5)u \qquad (5.103)$$

For the adjoint spinors

$$\bar{U}_L = \left[\frac{1}{2}(1-\gamma^5)u\right]^+ \gamma_0 = u^+ \frac{1}{2}(1-\gamma^5)\gamma_0 = u^+\gamma_0 \frac{1}{2}(1+\gamma^5) = \bar{u}\frac{1}{2}(1+\gamma^5) \qquad (5.104)$$

Similarly,

$$\bar{U}_R = \bar{u}\frac{1}{2}(1-\gamma^5) \qquad (5.105)$$

For the positron spinor $v(p)$, we have

$$\gamma^5 v(p) \simeq -\vec{\Sigma} \bullet \hat{p}\ v(p) \tag{5.106}$$

at the high-energy limit. The left- and right-handed spinors and their adjoints for the positrons are

$$\begin{aligned} v_R &= \frac{1}{2}\left(1-\gamma^5\right)v & v_L &= \frac{1}{2}\left(1+\gamma^5\right)v \\ \overline{v}_R &= \overline{v}\left(\frac{1}{2}\right)\left(1+\gamma^5\right) & \overline{v}_L &= \overline{v}\left(\frac{1}{2}\right)\left(1-\gamma^5\right) \end{aligned} \tag{5.107}$$

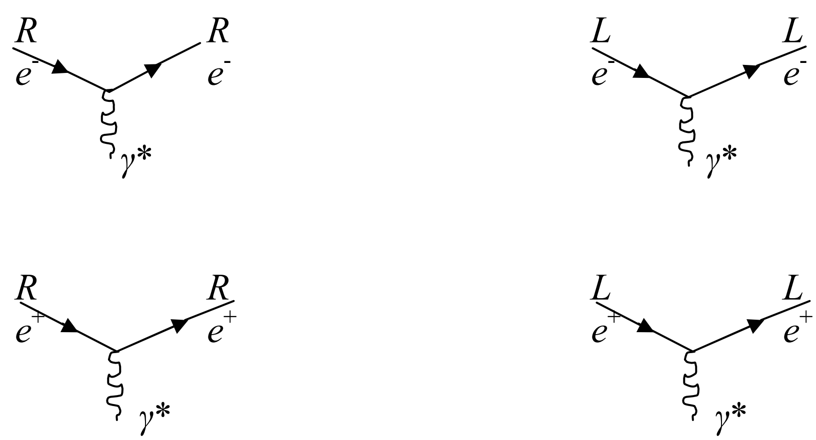
It is straightforward to show that

$$\begin{aligned} \overline{u}\gamma^\mu u &= \left(\overline{u}_R + \overline{u}_L\right)\gamma^\mu\left(u_R + u_L\right) \\ &= \overline{u}_R\gamma^\mu u_R + \overline{u}_L\gamma^\mu u_L \end{aligned} \tag{5.108}$$

and

$$\overline{v}\gamma^\mu v = \overline{v}_R\gamma^\mu v_R + \overline{v}_L\gamma^\mu v_L \tag{5.109}$$

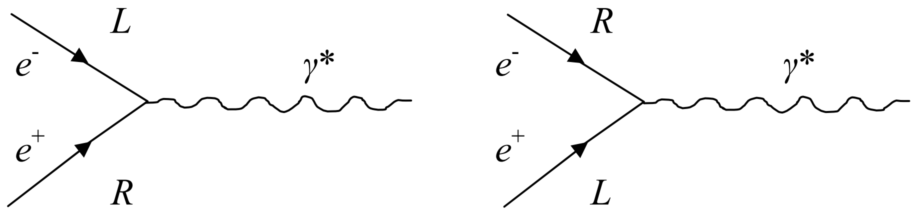
Equations 5.108 and 5.109 show that at high energy, the helicity of electron and positron is conserved in an electromagnetic interaction:



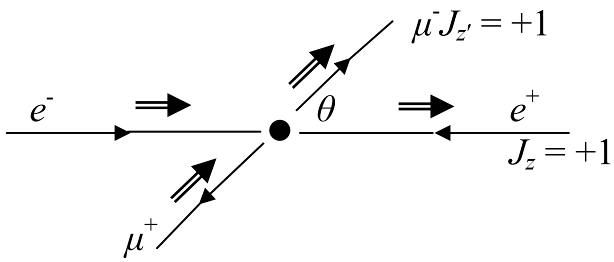
Similarly, one can readily show

$$\bar{v}\gamma^{\mu}u = \bar{v}_R\gamma^{\mu}u_L + \bar{v}_L\gamma^{\mu}u_R \tag{5.110}$$

Equation 5.110 implies



Helicity conservation is a very useful concept for understanding the angular distributions of many electromagnetic interactions. For example, consider the $e^- + e^- \rightarrow \mu^- + \mu^-$ scattering in the C.M. frame



One possible helicity structure (⇑ indicates the helicity of $e^{\mp}$, $\mu^{\mp}$) is shown here. For a given scattering angle θ, there are four possible combinations for J_z and $J_{z'}$, namely $J_z = \pm 1 \otimes J_{z'} = \pm 1$. The corresponding decay amplitude is $d^J_{\lambda'\lambda}(\theta)$, where λ, λ′ are the helicity of the virtual photon.

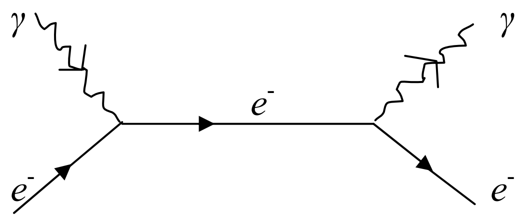
Since
$$d^1_{11}(\theta) = d^1_{-1-1}(\theta) = (1+\cos\theta)/2$$
$$d^1_{1-1}(\theta) = d^1_{-1+1}(\theta) = (1-\cos\theta)/2$$

one can show that the angular distribution is an average of $(1 + \cos\theta)^2$ and $(1 - \cos\theta)^2$, or $1 + \cos^2\theta$, as in Equation 5.85.

2.4 Compton Scattering and Pair Annihilation

As a final example for EM interaction involving spin-½ particle, we consider the Compton scattering $\gamma e^- \to \gamma e^-$ and the related pair annihilation $e^- e^+ \to \gamma\gamma$ process.

One diagram contributing to the Compton scattering is



Some new features are introduced in this diagram:
a) An external photon line is involved
b) An internal electron line (propagator) is involved

External Photons

We recall the Maxwell Equations:

$$\begin{aligned} \vec{\nabla}\bullet\vec{\mathrm{E}} &= \zeta \qquad \vec{\nabla}\times\vec{\mathrm{E}} + \frac{\partial\vec{B}}{\partial t} = 0 \\ \vec{\nabla}\bullet\vec{B} &= 0 \qquad \vec{\nabla}\times\vec{B} - \frac{\partial\vec{\mathrm{E}}}{\partial t} = \vec{j} \end{aligned} \tag{5.111}$$

The homogeneous equation $\vec{\nabla}\bullet\vec{B} = 0$ implies $\vec{B} = \vec{\nabla}\times\vec{A}$ and $\vec{\nabla}\times\vec{\mathrm{E}} + \frac{\partial\vec{B}}{\partial t} = 0$ implies $\vec{\nabla}\times\left(\vec{\mathrm{E}} + \frac{\partial\vec{A}}{\partial t}\right) = 0$ and $\vec{\mathrm{E}} + \frac{\partial\vec{A}}{\partial t} = -\vec{\nabla}\phi$.

Therefore, the homogeneous Maxwell Equations introduce the E.M. potential $A^\mu = \left(\phi, \vec{A}\right)$ from which the $\vec{\mathrm{E}}$ and $\vec{B}$ fields can be calculated:

$$\vec{B} = \vec{\nabla}\times\vec{A} \qquad \vec{E} = -\vec{\nabla}\phi - \frac{\partial\vec{A}}{\partial t} \tag{5.112}$$

The inhomogeneous Maxwell equations can be written as

$$\Box^2 A^\mu - \partial^\mu\left(\partial_\nu A^\nu\right) = j^\mu \tag{5.113}$$

where

$$j^\mu = \left(\zeta, \vec{j}\right)$$

(To obtain Equation 5.113, we substitute Equation 5.112 into $\vec{\nabla}\bullet\vec{\mathrm{E}} = \zeta$, for example. We then have

$$-\vec{\nabla}\bullet\vec{\nabla}\phi - \frac{\partial}{\partial t}\vec{\nabla}\bullet\vec{A} = \zeta$$

$$\Rightarrow \frac{\partial^2}{\partial t^2}\phi - \nabla^2\phi - \frac{\partial}{\partial t}\left(\frac{\partial}{\partial t}\phi\right) - \frac{\partial}{\partial t}\left(\vec{\nabla}\bullet\vec{A}\right) = \zeta$$

$$\Rightarrow \Box^2 A^0 - \partial^0\left(\partial_\nu A^\nu\right) = j^0$$

Similarly,

$$\vec{\nabla}\times\vec{B} - \frac{\partial\vec{E}}{\partial t} = \vec{j}$$

$$\Rightarrow \ \Box^2 A^k - \partial^k\left(\partial_\nu A^\nu\right) = j^k)$$

Equation 5.113 shows Maxwell equations satisfy charge conservation namely

$$\partial_\mu j^\mu = \Box^2\partial_\mu A^\mu - \Box^2\partial_\nu A^\nu = 0 \tag{5.114}$$

By introducing the Field Strength Tensor, defined as

$$F^{\mu\nu} = \Box^2\partial^\mu A^\nu - \partial^\nu A^\mu \tag{5.115}$$

the Maxwell equation can be expressed as

$$\partial_\mu F^{\mu\nu} = j^\nu \tag{5.116}$$

$$F^{\mu\nu} = \begin{pmatrix} 0 & -Ex & -Ey & -Ez \\ Ex & 0 & -Bz & By \\ Ey & Bz & 0 & -Bx \\ Ez & -By & Bx & 0 \end{pmatrix} \tag{5.117}$$

For a given E and B, the potential A^μ is not uniquely determined:

$$A^{\mu\prime} = A^\mu + \partial^\mu \chi \tag{5.118}$$

where χ is any function, then

$$\begin{aligned} \vec{B}' &= \vec{\nabla}\times\vec{A}' = \vec{\nabla}\times\vec{A} + \vec{\nabla}\times\nabla\chi = \vec{\nabla}\times\vec{A} = \vec{B} \\ \vec{E}' &= -\frac{\partial\vec{A}'}{\partial t} - \vec{\nabla}\phi' = -\frac{\partial\vec{A}}{\partial t} - \vec{\nabla}\phi = \vec{E} \end{aligned} \tag{5.119}$$

The freedom to choose χ allows us to simplify Equation 5.114 further by requiring

$$\partial_\mu A^\mu = 0 \quad \text{(Lorentz condition)} \tag{5.120}$$

and Equation 5.114 becomes

$$\Box^2 A^\mu = j^\mu \tag{5.121}$$

For a free photon, $j^\mu = 0$, and we have

$$\Box^2 A^\mu = 0 \tag{5.122}$$

Solutions for Equation 5.122 can be written as

$$A^\mu = N\varepsilon^\mu(\vec{q})e^{-iq\bullet x} \tag{5.123}$$

ε^μ is the polarization vector, and Equation 5.122 requires that $q^2 = 0$ for A^μ in Equation 5.123, showing that the photon is massless.

The Lorentz condition $\partial_\mu A^\mu = 0$ further implies

$$q_\mu \varepsilon^\mu = q\bullet\varepsilon = 0 \tag{5.124}$$

Equation 5.124 shows that ε^μ consists of only three independent parameters. It is interesting that one can take advantage of an additional gauge freedom to remove yet another parameter in ε^μ. One can make the following transformation for A^μ:

$$A^{\mu} \to A^{\mu\prime} = A^{\mu} + \partial^{\mu}\Lambda \tag{5.125}$$

Now, Λ, unlike χ in Equation 5.118, cannot be an arbitrary function of χ and it must satisfy

$$\Box^2 \Lambda = 0 \tag{5.126}$$

(since $\partial_{\mu} A^{\mu\prime} = 0$, as required by the Lorentz condition)

A possible choice of Λ is a plane wave along q^{μ}:

$$\Lambda = iaNe^{iq\bullet x} \qquad A^{\mu\prime} = N\left(\varepsilon^{\mu} + a_q^{\mu}\right)e^{-iq\bullet x} \tag{5.127}$$

Now,

$$\varepsilon^{\mu} \to \varepsilon^{\mu\prime} = \varepsilon^{\mu} + aq^{\mu} \tag{5.128}$$

and Equation 5.124 implies

$$q_{\mu}\varepsilon^{\mu\prime} = q_{\mu}\varepsilon^{\mu} + aq^2 = q\bullet\varepsilon = 0$$

Equation 5.128 shows that one can select a value for the parameter a to make $\varepsilon^{0\prime}$ vanish; $\varepsilon^{0\prime} = 0$

The additional gauge freedom, Equation 5.125, therefore, reduces the number of independent parameters for ε^{μ} to 2. If we define the coordinates such that the free photon is moving along the z-axis, namely,

$$q^{\mu} = \left(q^0;\ 0,\ 0,\ q^0\right) \tag{5.129}$$

Then the conditions $q_{\mu}\varepsilon^{\mu} = 0$ and $\varepsilon^0 = 0$ imply that

$$\vec{q}\bullet\vec{\varepsilon} = 0$$

and ε^{μ} must be pointing in the plane perpendicular to photon direction of motion. In other words, the photons are transversely polarized. For linearly transversely polarized photons, we have

$$\varepsilon_1 = (0, 1, 0, 0); \;\; \varepsilon_2 = (0, 0, 1, 0)$$

and for circularly polarized photons, we have

$$\begin{aligned} \varepsilon_R &= \varepsilon(\lambda = +1) = -\sqrt{\frac{1}{2}}(0,\ 1,\ i,\ 0) \\ \varepsilon_L &= \varepsilon(\lambda = -1) = \sqrt{\frac{1}{2}}(0,\ 1,\ -i,\ 0) \end{aligned} \qquad (5.130)$$

The circularly polarized photons satisfy the

orthogonality relation: $\varepsilon^*(\lambda)\bullet\varepsilon(\lambda') = -\delta_{\lambda\lambda'}$ (5.131)

and

completeness relation: $\sum_{\lambda=R,L} (\varepsilon_\lambda)_i^* (\varepsilon_\lambda)_j = \delta_{ij} - \hat{q}_i \hat{q}_j$ (5.132)

We now turn to the issue of electron propagator.

We recall that for the Klein-Gordon equation, e.g.

$$\left(\Box^2 + m^2\right)\phi = -v\phi$$

the corresponding propagator for a spin-0 particle is

$$\frac{i}{p^2 - m^2} \qquad (5.133)$$

Analogously, for the Dirac equation

$$\left(\not{p} - m\right)\psi = -e\gamma^\mu A_\mu \psi$$

the propagator for spin-½ particle is

$$\frac{i}{\not{p} - m} = \frac{i\left(\not{p} + m\right)}{p^2 - m^2} = \frac{i\sum_s u^s \overline{u}^s}{p^2 - m^2} \qquad (5.134)$$

The summation over the spins in Equation 5.134 is a general feature for propagators with spins.

Recall that the photon propagator

$$i\frac{-g_{\mu\nu}}{q^2} = \frac{i\sum_{\lambda=1}^{4}\varepsilon_\mu^{*(\lambda)}\varepsilon_\nu^{(\lambda)}}{q^2} \tag{5.135}$$

where

$$\sum_{\lambda=1}^{4}\varepsilon_\mu^{*(\lambda)}\varepsilon_\nu^{(\lambda)} = \sum_T \varepsilon_\mu^{T*}\varepsilon_\nu^T + \varepsilon_\mu^{L*}\varepsilon_\nu^L + \varepsilon_\mu^{S*}\varepsilon_\nu^S \tag{5.136}$$

where T, L, S signifies Transverse, Longitudinal, and Scalar (time-like) polarizations for the virtual photons.

For a massive vector boson ($w^\pm$, z^0), the corresponding equation is

$$\left(\Box^2 + M^2\right)B^\nu - \partial^\mu\left(\partial_\nu B^\nu\right) = 0 \tag{5.137}$$

The corresponding propagator is

$$\frac{i\left(-g^{\mu\nu} + p^\mu p^\nu / M^2\right)}{p^2 - M^2} = \frac{i\sum_\lambda \varepsilon_\mu^{(\lambda)*}\varepsilon_\nu^{(\lambda)}}{p^2 - M^2} \tag{5.138}$$

There are three helicity states for massive spin-1 bosons

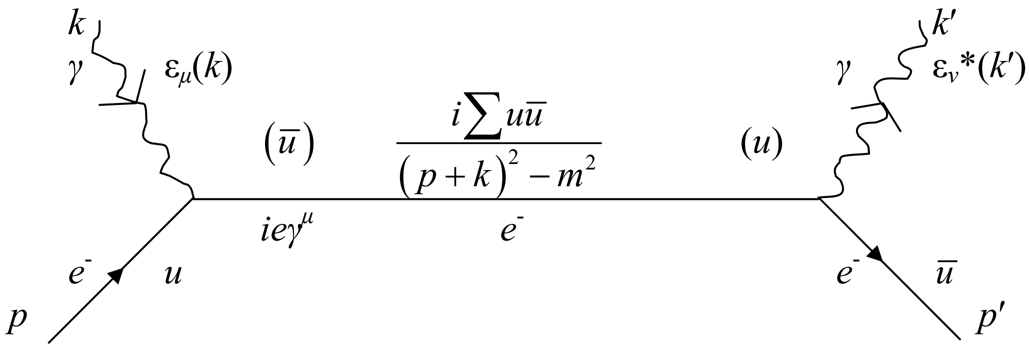
$$\begin{aligned} \varepsilon^{(\lambda=\pm1)} &= \mp\left(0,\ 1,\ \pm i,\ 0\right)\big/\sqrt{2} \\ \varepsilon^{(\lambda=0)} &= \left(|p|,\ 0,\ 0,\ E\right)\big/M \end{aligned} \tag{5.139}$$

The completeness relation $\displaystyle\sum_\lambda \varepsilon_\mu^{(\lambda)*}\varepsilon_\nu^{(\lambda)} = -g_{\mu\nu} + \frac{p_\mu p_\nu}{M^2}$ (5.140)

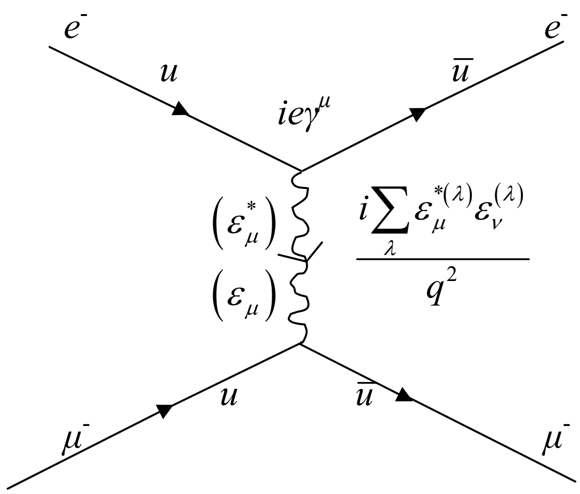
is used in Equation 5.138 to express the propagator in terms of the boson's polarization vectors.

Now, we are ready to consider the Compton scattering.

First, the Feynman diagram for the Compton scattering is

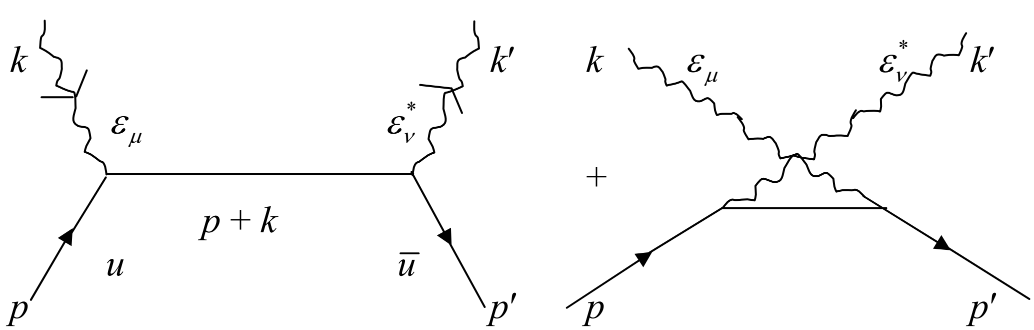


Comparing with the $e^- \mu^- \rightarrow e^- \mu^-$ diagram, which has internal photon propagator and external electron (rather than internal electron propagator and external photon),



the $ee\gamma$ vertices has the form $\bar{u}\varepsilon_\mu^*\left(ie\gamma^\mu\right)u$ in both diagrams. This similarity is manifested only after the propagator is expressed in terms of the spin-sum.

There are two diagrams contributing to the $\gamma e^- \rightarrow \gamma e^-$ Compton scattering



The invariant scattering amplitudes for the two diagrams are

$$-im_1 = \bar{u}\varepsilon_\nu^* \left(ie\gamma^\nu\right) \frac{i\left(\not{p} + \not{k} + m\right)}{\left(p+k\right)^2 - m^2} \left(ie\gamma^\mu\right) \varepsilon_\mu u \tag{5.141}$$

and

$$-im_2 = \bar{u}\varepsilon_\mu \left(ie\gamma^\mu\right) \frac{i\left(\not{p} - \not{k}' + m\right)}{\left(p-\not{k}'\right)^2 - m^2} \left(ie\gamma^\nu\right) \varepsilon_\nu^* u \tag{5.142}$$

m_2 is obtained from m_1 by

$$k \leftrightarrow -k' \qquad \varepsilon_\mu \leftrightarrow \varepsilon_\nu^*$$

There is an interesting implication for gauge invariance. Since $\varepsilon_\mu \to \varepsilon_\mu + ak_\mu$ should not change physics. m should be invariant under this transformation. This implies that replacing ε_μ by k_μ in m, m should vanish. i.e.:

$$m\left(\varepsilon_\mu \to k_\mu\right) = 0 \tag{5.143}$$

Similarly

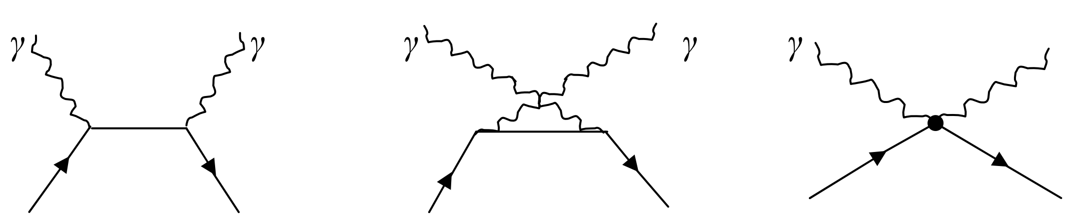
$$m\left(\varepsilon_\nu \to k'_\nu\right) = 0$$

It can be readily verified that $m_1\left(\varepsilon_\mu \to k_\mu\right) \neq 0$.

Similarly, $m_2\left(\varepsilon_\mu \to k_\mu\right) \neq 0$.

However, $m = m_1 + m_2$ satisfies $m\left(\varepsilon_\mu \to k_\mu\right) = 0$.

It is interesting to note that Compton scattering for a spin-0 particle, $\gamma\pi^+ \to \gamma\pi^+$, contains three diagrams:



π^+ π^+ π^+ π^+ π^+ π^+

The third diagram, the contact interaction, has its origin in the e^2A^2 term:

$$V_{K-G} = -ie\left(\partial_\mu A^\mu + A^\mu \partial_\mu\right) - e^2 A^2 \tag{5.144}$$

The diagram for the contract interaction cannot be ignored. In fact, one can show that the gauge-invariance $m\left(\varepsilon_\mu \to k_\mu\right) = 0$ cannot be satisfied by the first two diagrams alone. It is only satisfied after including the third diagram:

$$M = M_1 + M_2 + M_3$$

Coming back to the $\gamma e^- \to \gamma e^-$ Compton scattering, we can evaluate m_1 by ignoring the electron mass. At this high-energy limit, we have

$$m_1 = \varepsilon_\nu^* \varepsilon_\mu e^2 \bar{u}\gamma^\nu \left(\not{p} + \not{k}\right)\gamma^\mu u/s \tag{5.145}$$

Summing over the photon polarization and electron spin, and averaging over the initial spins, we have

$$\overline{\left|m_1\right|^2} = \frac{e^4}{4s^2}\sum_{s,s'} \bar{u}^{s'}\gamma^\nu \left(\not{p} + \not{k}\right)\gamma^\mu u^s \left(\bar{u}^s \gamma_\mu \left(\not{p} + \not{k}\right)\gamma_\nu u^{s'}\right) \tag{5.146}$$

where we use

$$\sum_\lambda \varepsilon_\mu^* \varepsilon_{\mu'} = -g_{\mu\mu'}$$

Equation 5.146 can be evaluated using trace theorems and contraction theorems:

$$\begin{aligned} \overline{\left|m_1\right|^2} &= \frac{e^4}{4s^2} Tr\left(\underbrace{\not{p}'\gamma^\nu}_{-2\not{p}'}\left(\not{p} + \not{k}\right)\underbrace{\gamma^\mu \not{p}\gamma_\mu}_{-2\not{p}}\left(\not{p} + \not{k}\right)\gamma_\nu\right) \\ &= \frac{e^4}{s^2} Tr\left(\not{p}'\left(\not{p} + \not{k}\right)\not{p}\left(\not{p} + \not{k}\right)\right) \end{aligned} \tag{5.147}$$

The only term which survives in Equation 5.147 is

$$\overline{|m_1|^2} = \frac{e^4}{s^2} Tr\left(\not{p}' \not{k} \not{p} \not{k} \right)$$
$$= \frac{8e^4}{s^2} (p'\bullet k)(p\bullet k) = 2e^4 \left(-\frac{u}{s} \right) \tag{5.148}$$
$$\left(p\bullet k = \frac{s}{2}, \quad p'\bullet k = -\frac{u}{2} \right)$$

$\overline{|m_2|^2}$ can be obtained from $\overline{|m_1|^2}$ by $s \leftrightarrow u$ interchange. Therefore,

$$\overline{|m_2|^2} = \overline{|m_1|^2} (s \leftrightarrow u) = 2e^4 \left(-\frac{s}{u} \right) \tag{5.149}$$

The interference term can be shown to vanish. Hence, we have

$$\overline{|m|^2} = 2e^4 \left(-\frac{u}{s} - \frac{s}{u} \right) \tag{5.150}$$

For the related pair annihilation process

$$e^+ e^- \to \gamma\gamma$$

the scattering amplitude is obtained from the

$$\gamma e^- \to \gamma e^-$$

process by

$$k \leftrightarrow p'$$

At the high energy limit, the spin-averaged rate for $e^+ e^- \to \gamma\gamma$ is

$$\overline{|m|^2} = 2e^4 \left(\frac{u}{t} + \frac{t}{u} \right)$$

This concludes our discussion on Quantum Electro Dynamics.